Question 1

(a) $\dot{x} = y$

$\dot{y} = F(x,y)$ where $F(x,y) = \begin{cases} -x & |x| > \alpha \\ -x-y & |x| < \alpha, |y| < \alpha \\ 0 & \text{otherwise} \end{cases}$

[2]

Autonomous - time, t, does not appear explicitly in the governing equations. [1]

(b) critical point $\dot{x} = \dot{y} = 0$ $\Rightarrow y = 0$

$F(x,0) = 0 \Rightarrow x = 0$

critical point at $(0,0)$. [2]

consider eigenvalues of $\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ [1]

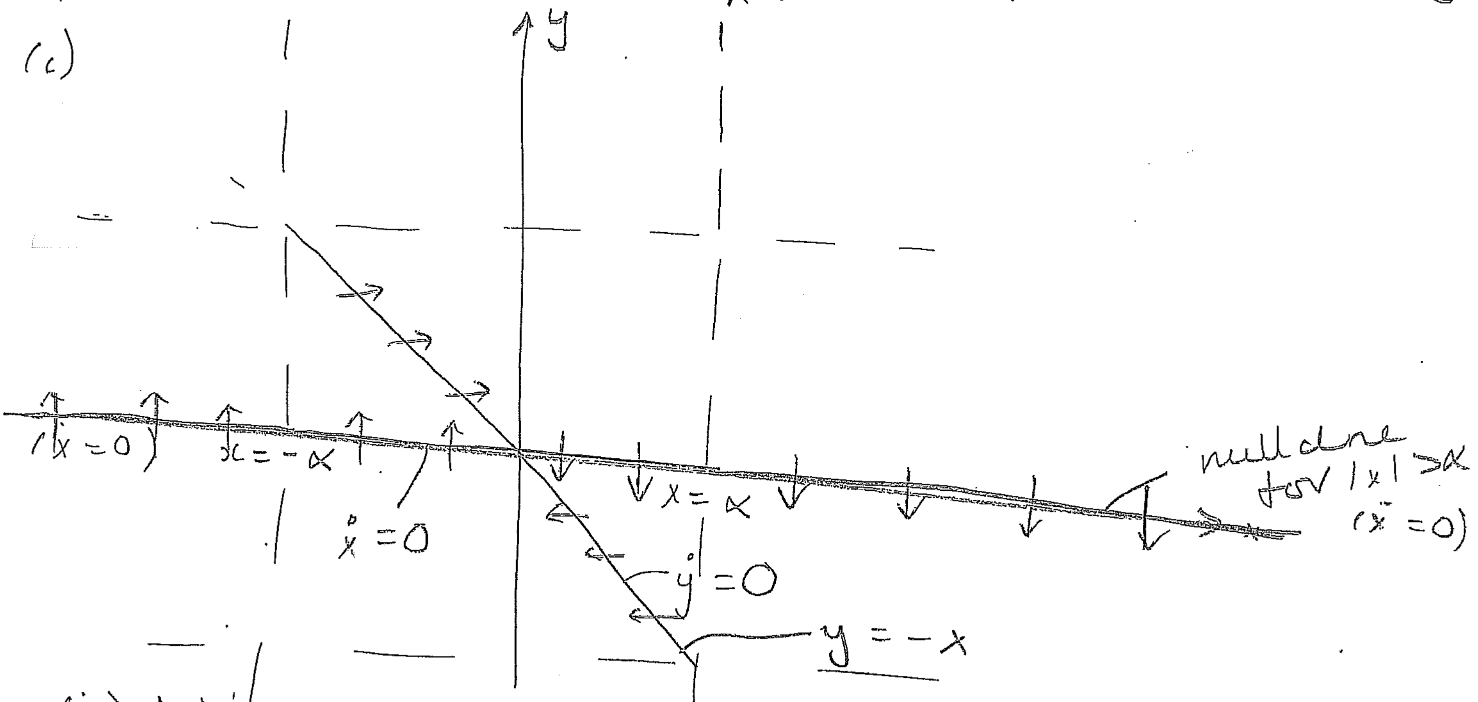
which are $-\lambda(-1-\lambda)+1 = 0$

$\lambda^2 + \lambda + 1 = 0$

$\lambda = \dfrac{-1 \pm \sqrt{1-4}}{2} = \dfrac{-1 \pm \sqrt{3}\,i}{2}$ [1]

critical point is a stable clockwise spiral. [2]

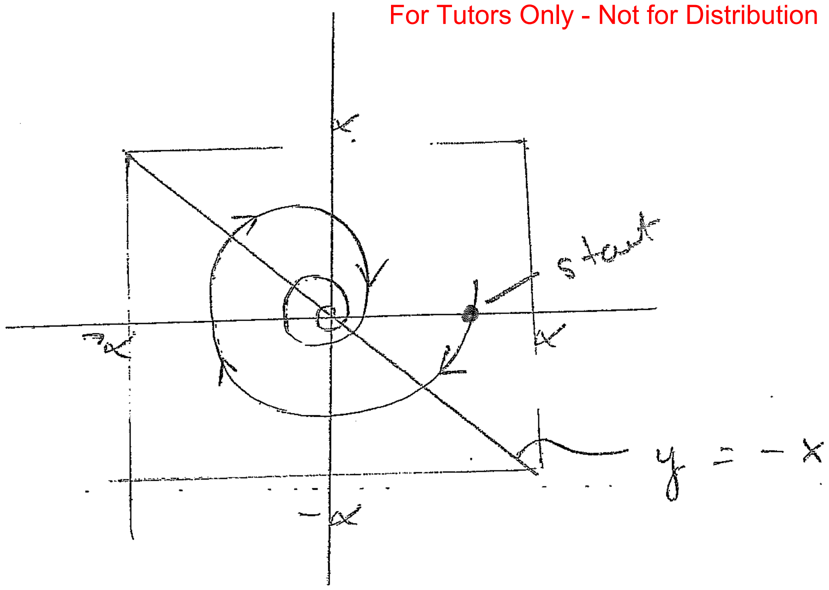
(c)



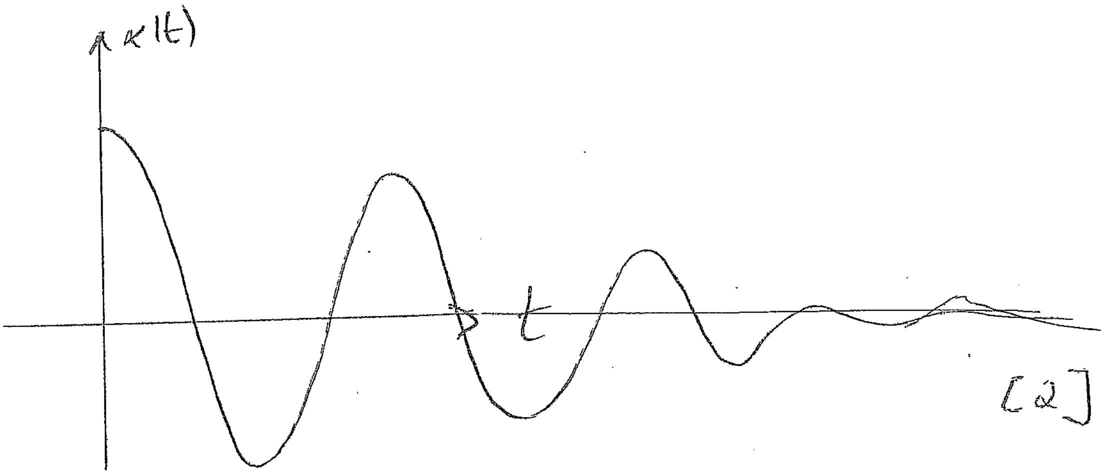
(i) $|x| > \alpha$ Null clines correspond to $y = 0$ [2]

(ii) $|x| < \alpha$, $|y| < \alpha$: Null clines are again $y = 0$ and the line $x + y = 0$ ie $y = -x$ [4]

d)



[2]



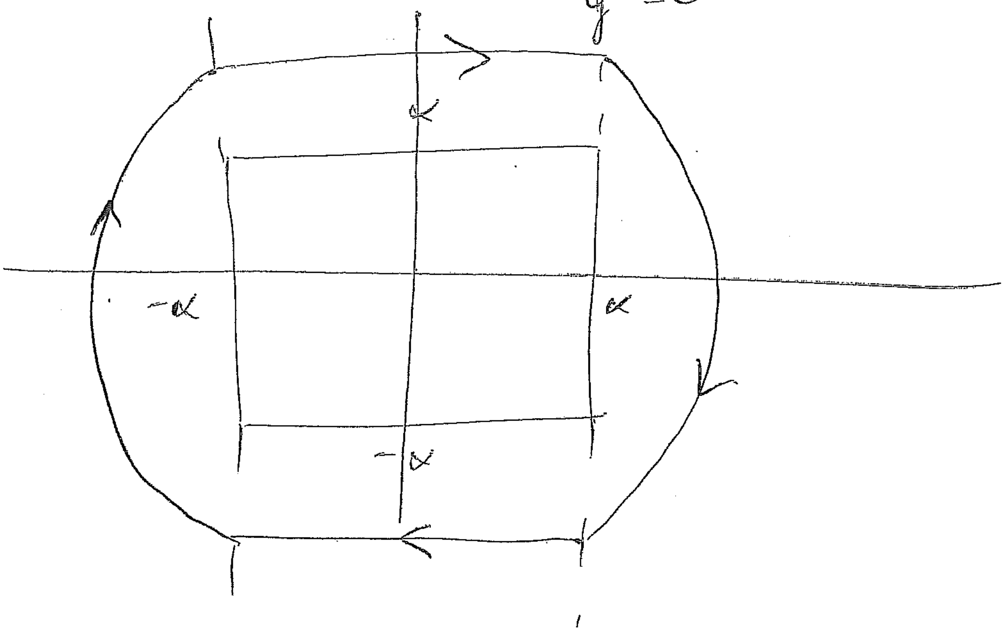
[2]

e) $|x| > \alpha$

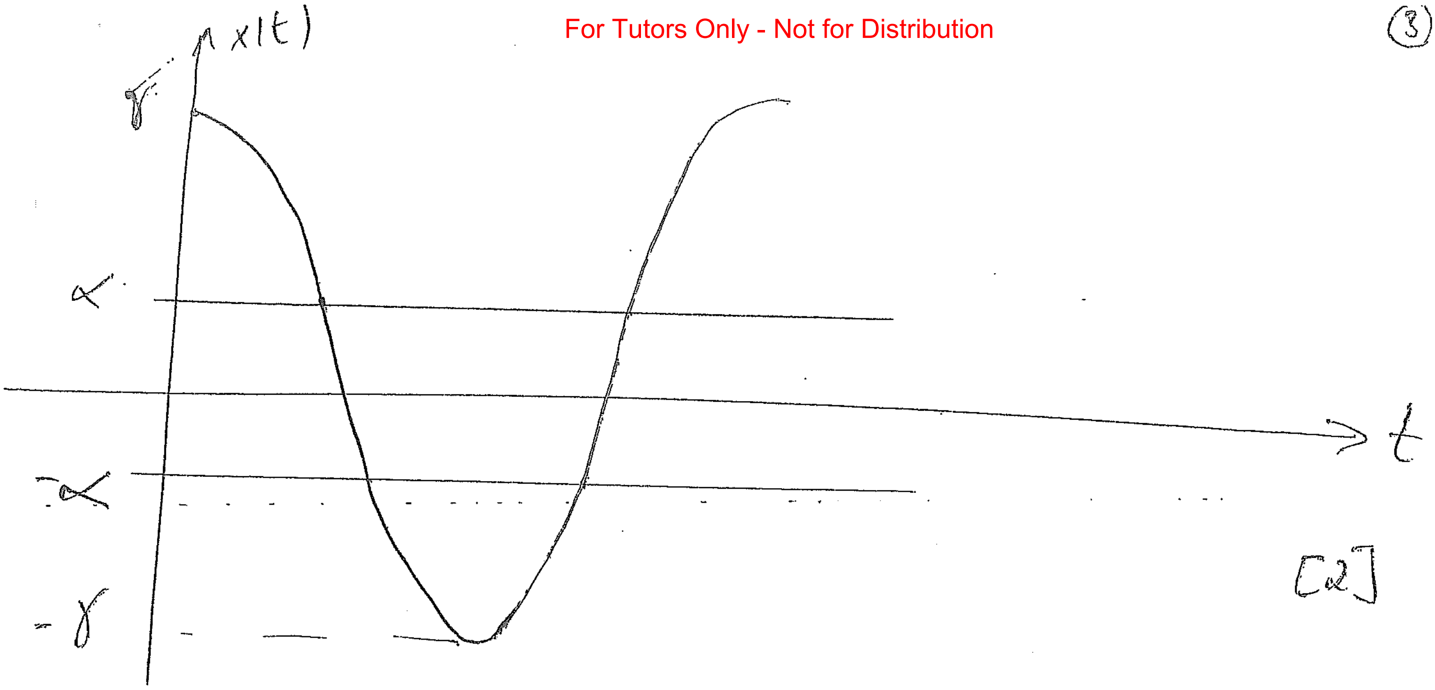
$\dot{x} = y$
$\dot{y} = -x$

$\Rightarrow \quad x^2 + y^2 = \text{const.}$ Trajectories are circular arcs. [1]

$|x| < \alpha, |\dot{x}| > \alpha \quad \Rightarrow \quad \dot{x} = y$, $\dot{y} = 0$ Trajectories are straight horizontal lines [1]



[2]



[2]

2 (a) $y'(x) = f(x, y(x)), y(a) = b$ has a solution in the rectangle $R := \{(x,y) : |x-a| \le h, |y-b| \le k\}$ provided:

P(i): (a) f is continuous in R, with bound M (so $|f(x,y)| \le M$) and (b) $Mh \le k$.

P(ii): f satisfies a Lipschitz condition in R so that there exists a real positive L such that $|f(x,u) - f(x,v)| \le L|u-v|$ for $(x,u) \in R, (x,v) \in R$.

Furthermore this solution is unique.

[4]

b) Gronwall's inequality:

Suppose $A \ge 0$ and $b \ge 0$ are constants and v is a non-negative cts function satisfying

$$v(x) \le b + A\left|\int_a^x v(s)\,ds\right|$$

Then $v(x) \le b e^{A|x-a|}$.

Suppose now that y and z are solutions of the ode $y'(x) = f(x, y(x))$ with $y(a) = b$ and $z(a) = c$, where f satisfies P(i) and P(ii).

Then $y(x) - z(x) = b - c + \int_a^x (f(s,y(s)) - f(s,z(s)))\,ds$

so that $|y(x) - z(x)| \le |b-c| + \left|\int_a^x |f(s,y(s)) - f(s,z(s))|\,ds\right|$

$$\le |b-c| + \left|\int_a^x L|y(s) - z(s)|\,ds\right|$$

and by Gronwall's inequality

$$|y(x) - z(x)| \le |b-c|\, e^{L|x-a|} \le |b-c|\, e^{Lh}.$$

Solution is ctsly dependent on the initial data if we can make $|y(x) - z(x)|$ as small as we like by taking $|b-c|$ small enough.

i.e. $\forall x \in [a-h, a+h]$ $\forall \varepsilon > 0$ $\exists \delta > 0$ st

$$|b-c| < \delta \Rightarrow |y(x) - z(x)| \le \varepsilon, \quad \forall x \in [a-h, a+h]$$

[~~6~~5].

Take $\delta = e^{-Lh}\varepsilon$ and result follows.

c) $f(x,y) = \sqrt{|x(1-y(x)^2)|}$ $\quad x \ge 0$.

$y(x) = 1$, $f(x,y) = 0$

for $y(x) < 1$ $\quad f(x,y) = \sqrt{x(1-y^2)} > 0$ $\quad$ y is increasing

$y(x) > 1$ $\quad f(x,y) = \sqrt{x(y^2-1)} > 0$ $\quad$ "

[~~1~~ 2]

We need y bounded away from 1, so take $k < 1$ (2)

Then $f(x,y)$ is continuous on R and

$$M = \sup_R |x(1-y(x))^2|^{1/2} \leq h^{1/2}.$$

We need $Mh \leq k$ ie $h^{3/2} \leq k$

So $P(i)$ holds for $h \leq k^{2/3}$. [3]

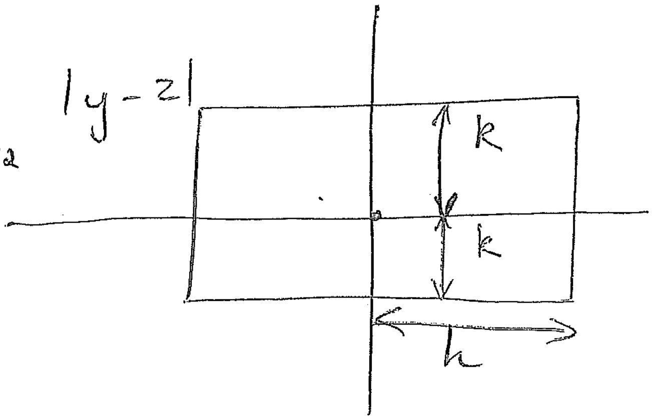
Also, by the MVT there exists ξ between y and z such that

$$|f(x,y) - f(x,z)| = |f_y(x,\xi)||y-z|$$

$$= |x|^{1/2}\left|\frac{1}{\cancel{2}}(1-\xi^2)^{-1/2} \cdot -\cancel{2}\xi\right| |y-z|$$

$$= \frac{|x|^{1/2}}{|(1-\xi^2)^{1/2}|} |\xi| \, |y-z|$$

$$\leq \frac{h^{1/2}k}{(1-k^2)^{1/2}} |y-z|$$



So $P(ii)$ holds with

$$L = \frac{h^{1/2}k}{(1-k^2)^{1/2}}.$$

Thus there is a unique solution in R for $h \leq k^{2/3}$. [3]

$$\frac{dy}{dx} = x^{1/2}(1-y(x)^2)^{1/2} \qquad y(0) = 0$$

$$\int \frac{1}{\sqrt{(1-y^2)}} dy = \int x^{1/2} dx$$

$y = \sin t$

$$y = \sin\left(\frac{2}{3}x^{2/3} + c\right)$$

$y(0) = 0 \Rightarrow c = 0.$

$$y(x) = \sin\left(\frac{2}{3}x^{3/2}\right) \qquad [2]$$ 5

(iii) $y(0) = 1$.

Clearly $y(x) = 1$ is a solution.

y increasing so $\dfrac{dy}{dx} = \sqrt{x(y^2-1)}$

$$\int \frac{1}{\sqrt{y^2-1}}\,dy = x^{1/2}\,dx$$

$y = \cosh t$

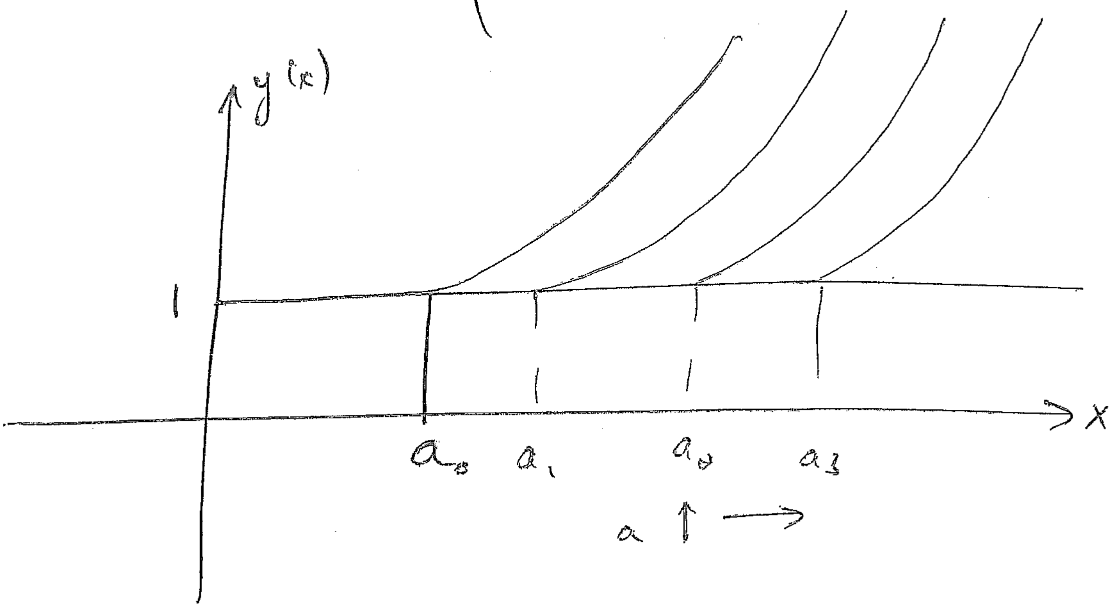
$$y = \cosh^{-1}\left(\frac{2}{3}x^{3/2} + \text{const}\right)$$

suppose $y = 1$ at $x = a$.

$$1 = \cosh^{-1}\left(\frac{2}{3}a^{3/2} + \text{const}\right)$$

$$\text{const} = \cosh 1 - \frac{2}{3}a^{3/2}$$

$$y(x) = \cosh^{-1}\left(\frac{2}{3}x^{3/2} - \frac{2}{3}a^{3/2} + \cosh 1\right)$$



[4]

Infinitely many solutions as $f(x,y)$ not Lipschitz at $y=1$.

Hence does not contradict uniqueness result. [2]

Question 5

(a) Suppose $\Gamma = (x(t), y(t), z(t))$ in terms of a parameter t.
The characteristic equations are

$$\frac{dx}{dt} = -y \; ; \; \frac{dy}{dt} = x, \; \frac{dz}{dt} = 2xyz \qquad [2]$$

The curve Γ is a characteristic curve [2]

The curve $(x(t), y(t), 0)$ which lies below the characteristic curve in the (x,y)-plane is called a characteristic projection. [2]

(b) $$x\frac{dx}{dt} = -xy$$
$$y\frac{dy}{dt} = xy \quad \Rightarrow \quad x^2 + y^2 = \text{const} \quad \text{are characteristic ~~curves~~ projections.}$$

These are circles centre the origin in the (x,y)-plane. [2]

consider $$\frac{d}{dt}\left(z e^{x^2}\right) = e^{x^2}\frac{dz}{dt} + 2xze^{x^2}\frac{dx}{dt}$$
$$= e^{x^2} 2xy + 2xze^{x^2}\cdot -y = 0$$

$\Rightarrow$ ze^{x^2} is constant on each characteristic curve. [2]

(c) Consider the initial data, and parametrise by s
$(s, 1-s^2, s^2e^{-s^2})$.

So characteristic curves are

$$x^2+y^2 = s^2 + (1-s^2)^2 \qquad \tfrac{1}{2} \le s \le 1$$
$$x(t) = \sqrt{s^2+(1-s^2)^2}\,\sin t$$
$$y(t) = \sqrt{s^2+(1-s^2)^2}\,\cos t$$

$$J = \begin{vmatrix} x_s & x_t \\ y_s & y_t \end{vmatrix} = \begin{vmatrix} \frac{1}{2}(\;)^{-1/2}\cdot(2s+2(1-s^2)\cdot -2s)\sin t & \sqrt{\;}\cos t \\ \frac{1}{2}(\;)^{-1/2}(2s+2(1-s^2)\cdot -2s)\cos t & -\sqrt{\;}\sin t \end{vmatrix}$$

$$s - 2s(1-s^2)$$

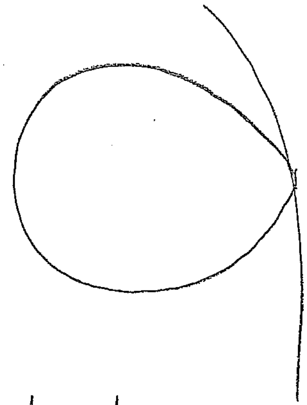
$J = 0$ when $s(-1+2s^2) = 0$

$$s = 0 \text{ or } s = \pm\frac{1}{\sqrt{2}}$$

So data is cauchy for $\frac{1}{2} \leq s < \frac{1}{\sqrt{2}}$

$\frac{1}{\sqrt{2}} < s \leq 1.$ [4]

At $s = \frac{1}{\sqrt{2}}$, characteristic projections and data curve touch.



i) $Z \exp x^2$ constant on characteristics.

For $\frac{1}{2} \leq s < \frac{1}{\sqrt{2}}$, characteristics are circles of radius between $\sqrt{\frac{1}{4} + \left(1-\frac{1}{4}\right)^2}$

and $\sqrt{\frac{1}{2} + \left(1-\frac{1}{2}\right)^2}$ ie $\sqrt{\frac{1}{4} + \frac{9}{16}} = \sqrt{\frac{13}{16}}$

and $\sqrt{\frac{1}{2} + \frac{1}{4}} = \sqrt{\frac{3}{4}}$

circles of radius $\sqrt{\frac{13}{16}}$ to $\sqrt{\frac{3}{4}}$

For $\frac{1}{\sqrt{2}} < s \leq 1$ circles of radius between $\sqrt{\frac{3}{4}}$ and 1.

$\frac{1}{2} \leq s < \frac{1}{\sqrt{2}}$ $\quad \frac{13}{16} \leq x^2 + y^2 < \frac{3}{4}$

$\frac{1}{\sqrt{2}} < s \leq 1 \Rightarrow \frac{3}{4} < x^2 + y^2 \leq 1$

$$Z e^{x^2} = s^2 e^{-s^2} . e^{s^2} = s^2$$

$$x^2 + y^2 = s^2 + (1-s^2)^2$$

So $s^2 + 1 - 2s^2 + s^4 = x^2 + y^2$

$$s^4 - s^2 + 1 - (x^2+y^2) = 0$$

$$s^2 = \frac{1 \pm \sqrt{1 - 4(1-(x^2+y^2))}}{2}$$

$$= \frac{1 \pm \sqrt{4(x^2+y^2) - 3}}{2}$$

$$x^2+y^2 \geq \frac{13}{16}$$

$$4(x^2+y^2) - 3 \geq \frac{13}{4} - 3 > 0$$

So real roots.

$x^2+y^2 \leq 1 \Rightarrow 4(x^2+y^2) - 3 \leq 1$ so both s^2 +ive.

$$\text{So } s = \pm\sqrt{\left(\frac{1 \pm \sqrt{4(x^2+y^2) - 3}}{2}\right)}$$

Take +ive roots.

For $\frac{13}{16} \leq x^2+y^2 < \frac{3}{4}$ (ie $\frac{1}{2} \leq s < \frac{1}{\sqrt{2}}$)

have $s = \dfrac{1 - \sqrt{4(x^2+y^2) - 3}}{2}$

For $\frac{3}{4} < x^2+y^2 \leq 1$ (ie $\frac{1}{\sqrt{2}} < s \leq 1$)

$$s = \sqrt{\frac{1 + \sqrt{4(x^2+y^2) - 3}}{2}}$$

So

$$z = \left[\frac{\left(1 - \sqrt{4(x^2+y^2)-3}\right)}{2}\right]^{2} e^{-x^2} \qquad \frac{13}{16} \leq x^2+y^2 < \frac{3}{4}$$

$$z = \left[\frac{\left(1 + \sqrt{4(x^2+y^2)-3}\right)}{2}\right]^{2} e^{-x^2} \qquad \frac{3}{4} < x^2+y^2 \leq 1$$

[9]