DEGREE OF MASTER OF SCIENCE

MATHEMATICAL MODELLING AND SCIENTIFIC COMPUTING

A2 Mathematical Methods II

HILARY TERM 2018

THURSDAY, 19 APRIL 2018, 9.30am to 11.30am

This exam paper contains three sections.
Candidates should submit answers to a maximum of ***four*** *questions for credit that include an answer to at least* ***one*** *question in each section.*

Please start the answer to each question in a new answer booklet.
All questions will carry equal marks.

Do not turn this page until you are told that you may do so

Section A: Nonlinear Systems

1. [25 marks]

   (a) [5 marks] Consider a system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with $\mathbf{x} \in E \subset \mathbb{R}^n$, $\mathbf{f} \in C^1(E)$, and assume that it has a fixed point $\mathbf{x}_0$. Define what is meant by a *Lyapunov function* for this system at the fixed point. State briefly (without proof) the relationship between Lyapunov functions and the stability of a fixed point.

   (b) [5 marks] Consider the system

$$\begin{cases} \dot{x} = -x + 2y^3 - 2y^4 \\ \dot{y} = -x - y + xy \end{cases}$$

   with $(x, y) \in \mathbb{R}^2$. Find a suitable choice for the constant a so that

$$V = x^2 + ay^4$$

   is a Lyapunov function for the system at the origin. Use this Lyapunov function to prove that the fixed point at the origin is asymptotically stable and that its domain of attraction is the entire plane.

   (c) [15 marks] Consider the system

$$\begin{cases} \dot{x} = y \\ \dot{y} = -f'(x) + \alpha y \end{cases}$$

   with $(x, y) \in \mathbb{R}^2$, $f \in C^2(\mathbb{R})$. Assume that $f(x)$ has a non-degenerate minimum at x_0.

   (i) For $\alpha > 0$, show that the fixed point $(x_0, 0)$ is unstable.

   (ii) For $\alpha = 0$, find a Lyapunov function and prove that the fixed point $(x_0, 0)$ is stable.

   (iii) For $\alpha < 0$, find a Lyapunov function and prove that the fixed point $(x_0, 0)$ is asymptotically stable.

2. [25 marks]

(a) [15 marks]

(i) [5 marks] Consider a system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with $\mathbf{x} \in E \subset \mathbb{R}^n$, $\mathbf{f} \in C^r(E)$, with $r \geqslant 1$, and assume that it has a fixed point $\mathbf{x}_0$. What does it mean for this fixed point to be *hyperbolic*? When does this fixed point have a non-empty *local centre manifold*? Explain how this manifold is defined.

(ii) [10 marks] Consider the system

$$\begin{cases} \dot{x} = y - x + xy \\ \dot{y} = x - y - x^2 \end{cases}$$

with $(x, y) \in \mathbb{R}^2$. Show that the origin is not a hyperbolic fixed point. Compute the lowest nonlinear approximation of the centre manifold and determine the stability of the origin.

(b) [10 marks]

(i) [5 marks] Consider a system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with $\mathbf{x} \in E \subset \mathbb{R}^n$, $\mathbf{f} \in C^1(E)$. What does it mean for a set $V \subset E$ to be an *invariant set*? When is an invariant set also an *attracting set*?

(ii) [5 marks] Consider the system

$$\begin{cases} \dot{x} = -y + x(1 - x^2 - y^2) \\ \dot{y} = x + y(1 - x^2 - y^2) \\ \dot{z} = 1 \end{cases}$$

with $(x, y, z) \in \mathbb{R}^3$. Show that the z-axis is an invariant set, but that it is not attracting. Prove the existence of a two-dimensional attracting set and specify its domain of attraction.

Section B: Further Mathematical Methods

3. (a) [10 marks] Consider the equation

$$\ddot{x}(t) + \mu x(t) = f(t), \qquad x(0) = x(2\pi), \quad \dot{x}(0) = \dot{x}(2\pi),$$

where $\mu \geqslant 0$ is constant, $f(t)$ is a given, continuous, real-valued function of t, and a dot represents $\mathrm{d}/\mathrm{d}t$. Use the Fredholm Alternative to determine under what conditions on μ and $f(t)$ there is

(i) a unique solution for $x(t)$;

(ii) no solution for $x(t)$;

(iii) multiple solutions for $x(t)$.

(b) Consider now the Mathieu equation

$$\ddot{x} + \mu x + \epsilon x \cos 2t = 0, \qquad x(0) = x(2\pi), \quad \dot{x}(0) = \dot{x}(2\pi),$$

where $0 < \epsilon \ll 1$. By expanding $x \sim x_0 + \epsilon x_1 + \cdots$ determine

(i) [8 marks] for which values of a and b there is a solution in which $\mu \sim a\epsilon + b\epsilon^2 + \cdots$;

(ii) [7 marks] for which values of c there is a solution in which $\mu \sim 1 + c\epsilon + \cdots$.

[*You may use the identities*

$$\cos t \cos 2t = \frac{\cos 3t + \cos t}{2}, \qquad \sin t \cos 2t = \frac{\sin 3t - \sin t}{2}, \qquad \cos^2 2t = \frac{1 + \cos 4t}{2},$$

without proof.]

4. (a) [10 marks] The function $y(x)$ minimises the functional

$$J = \int_0^1 F_1(x, y(x), y'(x))\,\mathrm{d}x + \int_1^2 F_2(x, y(x), y'(x))\,\mathrm{d}x$$

where F_1 and F_2 have continuous second partial derivatives and $y(x)$ is a continuous function, twice continuously differentiable in $0 \leqslant x < 1$ and in $1 < x \leqslant 2$, satisfying $y(0) = 0$ and $y(2) = 1$. Show that $y(x)$ satisfies the equations

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F_1}{\partial y'}\right) - \frac{\partial F_1}{\partial y} = 0 \qquad \text{for } 0 \leqslant x < 1,$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F_2}{\partial y'}\right) - \frac{\partial F_2}{\partial y} = 0 \qquad \text{for } 1 < x \leqslant 2,$$

with

$$\left.\frac{\partial F_1}{\partial y'}\right|_{x=1-} = \left.\frac{\partial F_2}{\partial y'}\right|_{x=1+}.$$

(b) [5 marks] A particle travels along the curve $y = y(x)$ from $y(0) = 0$ to $y(2) = 1$ with a speed equal to $v(x)$. Show that the time taken for the journey is

$$\int_0^2 \frac{\sqrt{1 + y'(x)^2}}{v(x)}\,\mathrm{d}x.$$

(c) [10 marks] Suppose now that

$$v(x) = \begin{cases} 1 & 0 \leqslant x < 1, \\ 2 & 1 \leqslant x \leqslant 2. \end{cases}$$

Show that the minimum journey time is

$$\frac{(4 - 3c)\sqrt{1 + c^2}}{2c},$$

where c satisfies

$$4(1 - c)^2(1 + c^2) - c^2(2 - 2c + c^2) = 0,$$

and find the corresponding path.

Section C: Further PDEs

5. (a) (i) [5 marks] Define the Hankel transform $\mathcal{H}[f(r);k]$ of a function $f(r)$. Show that

$$\mathcal{H}\left[\frac{\mathrm{d}^2 f}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}f}{\mathrm{d}r};k\right]=-k^2\mathcal{H}[f(r);k].$$

(ii) [9 marks] Let $c(x,y,t)$ satisfy the two-dimensional reaction-diffusion equation

$$\frac{\partial c}{\partial t}=\nabla^2 c-\alpha c, \qquad -\infty<x,y<\infty, \quad t\geqslant 0,$$

with $c\to 0$ as $x^2+y^2\to\infty$ and initial condition $c(x,y,0)=f(r)$ where $r=\sqrt{x^2+y^2}$, and

$$f(r)=\begin{cases} 0 & \text{for } r\leqslant a,\\ 1 & \text{for } a<r\leqslant b,\\ 0 & \text{for } r>b.\end{cases}$$

Show that

$$c(r,t)=\int_0^\infty [bJ_1(kb)-aJ_1(ka)]J_0(kr)\mathrm{e}^{-(k^2+\alpha)t}\,\mathrm{d}k.$$

[*Note that Bessel functions satisfy the differential equation*

$$J_n''(x)+\frac{1}{x}J_n'(x)+\left(1-\frac{n^2}{x^2}\right)J_n(x)=0,$$

and the recurrence relations

$$2J_n'(x)=J_{n-1}(x)-J_{n+1}(x), \qquad \frac{2n}{x}J_n(x)=J_{n-1}(x)+J_{n+1}(x).]$$

(b) The Mellin transform is given by

$$\mathcal{M}[f(x);s]=F(s)=\int_0^\infty x^{s-1}f(x)\,\mathrm{d}x,$$

which exists in some strip $c_1<\mathrm{Re}(s)<c_2$. The inversion is given by

$$f(x)=\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} x^{-s}F(s)\,\mathrm{d}s,$$

where $c_1<c<c_2$.

(i) [3 marks] Show that for $a>0$, $\mathcal{M}[f(ax);s]=a^{-s}\mathcal{M}[f(x);s]$.

(ii) [8 marks] For $x>0$, let

$$G(x)=\sum_{k=1}^\infty \frac{1}{1+k^2x^2}.$$

Show that

$$G(x)\sim\frac{\alpha}{x} \qquad \text{as } x\to 0+,$$

where you should determine the constant α.

[*You may use without proof the fact that*

$$\mathcal{M}\left[\frac{1}{1+x^2};s\right]=\frac{\pi}{2\sin(\pi s/2)} \qquad \text{for } 0<\mathrm{Re}(s)<2.$$

The Riemann zeta function, defined by

$$\zeta(x)=\sum_{n=1}^\infty n^{-x}$$

for Re(x) >1, may be analytically continued to a meromorphic function which has a single pole at $x=1$ with residue 1.]

6. (a) (i) [4 marks] Define the Fourier transform $\mathcal{F}[f(x);k] = \overline{f}(k)$ of a function $f(x) \in L^1$. If

$$h(x) = \int_{-\infty}^{\infty} f(y)g(x-y)\,\mathrm{d}y$$

show that

$$\overline{h}(k) = \overline{f}(k)\overline{g}(k).$$

(ii) [6 marks] Show that

$$\mathcal{F}\left[\frac{1}{1+x^2};k\right] = \pi \mathrm{e}^{-|k|}.$$

(iii) [4 marks] Solve the integral equation

$$u(x) + \int_{-\infty}^{\infty} \frac{u(y)}{1+(x-y)^2}\,\mathrm{d}y = f(x),$$

for $u(x)$ given $f(x)$.

(b) (i) [4 marks] By writing $z = r\mathrm{e}^{\mathrm{i}\theta}$ and evaluating the integral over θ, show that

$$\frac{1}{2\pi\mathrm{i}} \int_{|z|=r} z^{n-m-1}\,\mathrm{d}z = \delta_{nm} = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$$

(ii) [4 marks] The Z-transform of an ℓ^2 sequence $\{u_n\}$ is defined to be

$$U(z) = \sum_{n=-\infty}^{\infty} u_n z^n.$$

Use part (i) to deduce the inverse transform.

(iii) [3 marks] By applying the Z-transform to the difference equation

$$u_n - u_{n-1} = 1, \qquad n \geqslant 0,$$

with $u_n = 0$ for $n < 0$ deduce that

$$u_n = \frac{1}{2\pi\mathrm{i}} \int_{|z|=r} \frac{z^{-n-1}}{(1-z)^2}\,\mathrm{d}z.$$