A11475W1

DEGREE OF MASTER OF SCIENCE

Mathematical Modelling and Scientific Computing

Mathematical Methods II

Trinity Term 2023

Thursday, 20 April 2023, 2:30pm to 5.00pm

This exam paper contains three sections. You may attempt as many questions as you like but you must answer at least one question in each section. Your best answer in each section will count, along with your next best answer, making a total of four answers.

Candidates may bring a summary sheet into this exam consisting of (both sides of) one sheet of A4 paper containing material prepared in advance in accordance with the guidance given by the Mathematical Institute.

Please start the answer to each question in a new booklet.

Do not turn this page until you are told that you may do so

Nonlinear Systems

1. (a) [4 marks] Let $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{f} \in C^1(\mathbb{R}^n)$. Consider the system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \qquad (\dagger)$$

with associated flow φ_t. What does it mean to say that A is an attracting set of ($\dagger$)? What does it mean to say that $\mathbf{x}_0$ is a hyperbolic fixed point of saddle type? Suppose A is an attracting set, that $\mathbf{x}_0 \in A$ is a hyperbolic fixed point of saddle type, and $W^s(\mathbf{x}_0)$ and $W^u(\mathbf{x}_0)$ are the stable and unstable manifolds of $\mathbf{x}_0$ respectively. Must the following be true:

$$(1)\ W^s(\mathbf{x}_0) \subseteq A; \qquad (2)\ W^u(\mathbf{x}_0) \subseteq A?$$

(b) Consider the system

$$\begin{aligned} \dot{x} &= -2xy - 2x(x^2 + y^2 - 1), \\ \dot{y} &= 3x^2 + y^2 - 1. \end{aligned}$$

(i) [11 marks] Find the fixed points and determine their stable, unstable and centre linear subspaces.

(ii) [3 marks] Show that the line $L = \{(x, y) : x = 0\}$, the circle $C = \{(x, y) : x^2+y^2 = 1\}$, and the disk $D = \{(x, y) : x^2 + y^2 \leqslant 1\}$ are invariant sets.

(iii) [3 marks] Show that if $S = x(x^2 + y^2 - 1)$ then

$$\dot{S} = -2Sg(x, y),$$

where $g(x, y) > 0$ when $x^2 + y^2 > 1$.

(iv) [4 marks] Determine whether each of the following sets is an attracting set:

$$(1)\ L; \qquad (2)\ C; \qquad (3)\ D; \qquad (4)\ L \cup D.$$

2. Consider the system

$$\begin{aligned} \dot{x} &= \mu x + xz, \\ \dot{y} &= -y - z, \\ \dot{z} &= -2z - y^2 + x^2, \end{aligned}$$

where $\mu \in \mathbb{R}$ is a parameter.

(a) [4 marks] Find the fixed points, being careful to state for which values of μ each fixed point exists.

(b) [6 marks] Determine the stable, unstable and/or centre linear subspaces for the fixed point at the origin, being careful to consider all values of μ. For what value of μ is there a bifurcation of this fixed point?

(c) [8 marks] Find a quadratic approximation to the extended centre manifold in the vicinity of the origin.

(d) [2 marks] Determine the local dynamics on the extended centre manifold in the vicinity of the origin. Describe the type of bifurcation.

(e) [5 marks] For what other value of μ is there a bifurcation of some fixed point? Sketch the bifurcation diagram, plotting $x + y$ as a function of μ, including the stability of the branches near the origin. Explain why neither x, y or z individually are good variables to use when plotting the bifurcation diagram.

Further Mathematical Methods

3. (a) [6 marks] Consider the following linear equation for $u(x)$:

$$u'' + \lambda u = b(x), \quad 0 < x < \pi, \quad u(0) = u(\pi) = 0.$$

Here λ is a positive real parameter and $b(x)$ is given and square integrable on $[0, \pi]$. Use the Fredholm alternative to state the conditions under which this equation has either a unique solution or else has multiple linear solutions. In particular identify all of the critical values, $\lambda = \lambda^*$, at which there may be multiple solutions.

(b) [7 marks] Now consider the following nonlinear equation for $u(x)$:

$$u'' + \lambda u - u^3 = 0, \quad 0 < x < \pi, \quad u(0) = u(\pi) = 0.$$

Introduce a small parameter ϵ so that $\lambda = \lambda^* + \epsilon^2$, and consider small solutions as an asymptotic expansion

$$u = \epsilon\phi_1(x) + \epsilon^2\phi_2(x) + \epsilon^3\phi_3(x) + \dots$$

Using part (a), show that, for any choice of λ^*, we have $\phi_1(x) = \pm\frac{2}{\sqrt{3}}\sin\sqrt{\lambda^*}x$.

(c) [6 marks] Now for an odd integer $p \geqslant 3$ consider the following nonlinear equation for $u(x)$:

$$u'' + \lambda u - u^{p+1} = 0, \quad 0 < x < \pi, \quad u(0) = u(\pi) = 0.$$

Introduce a small parameter ϵ, and identify any small solutions, u, for λ close to λ^*.

(d) [6 marks] Let Ω be a bounded convex domain in $\mathbb{R}^m$ (where $m = 2, 3$), with a piecewise smooth boundary $\partial\Omega$.

Consider the operator L defined by the negative Laplacian on Ω together with homogeneous Dirichlet boundary conditions:

$$Lu = -\Delta u, \quad \mathbf{x} \in \Omega, \quad u(\mathbf{x}) = 0 \text{ for } \mathbf{x} \in \partial\Omega.$$

Show that L is self adjoint.

Assume that this problem has simple (real) eigenvalues $0 < \lambda_1 < \lambda_2 < \dots$, where each λ_k has a corresponding eigenfunction, denoted by $\psi_k(\mathbf{x})$.

Now consider

$$\Delta u + \lambda u = b(\mathbf{x}), \quad \mathbf{x} \in \Omega, \quad u(\mathbf{x}) = 0 \text{ for } \mathbf{x} \in \partial\Omega,$$

where $b(\mathbf{x})$ is assumed to be given and square integrable over Ω.

Use the Fredholm alternative to state the conditions under which this elliptic partial differential equation has either a unique solution or else has multiple linear solutions.

4. (a) [2 marks] State the Fundamental Lemma of the Calculus of Variations.

(b) [5 marks] Consider the functional

$$J[u] = \int_a^b F(x, u(x), u'(x))\,\mathrm{d}x,$$

where F is given and smooth, and u is in $C^2[a, b]$ such that $u(a) = c$ and $u(b) = d$. Show that at an extremal of J we must have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial u'}\right) - \frac{\partial F}{\partial u} = 0.$$

(c) [3 marks] Suppose that we relax the boundary condition imposed on u at $x = b$, and leave it unspecified. What is the natural boundary condition to be imposed when determining an extremal?

(d) [4 marks] Generalise part (b) to the case

$$J[u] = \int_a^b F(x, u(x), u'(x), u''(x), u'''(x))\,\mathrm{d}x,$$

where F is given and smooth, and y is in $C^4[a, b]$ and (u, u', u'') are all given at both $x = a, b$.

(e) [8 marks] Let Ω be a bounded convex domain in $\mathbb{R}^2$, with a piecewise smooth boundary denoted by $\partial\Omega$. Consider the functional, E:

$$E[(u, v)] = \int_\Omega F(x, y, u(x, y), u_x(x, y), u_y(x, y), v(x, y), v_x(x, y), v_y(x, y))\,\mathrm{d}x\mathrm{d}y,$$

defined for the C^2 vector field $(u(x, y), v(x, y)) : \Omega \to \mathbb{R}^2$, satisfying boundary conditions where (u, v) is given on $\partial\Omega$.

Deduce a pair of partial differential equations that the extremal (u, v) must satisfy.

Give the resulting partial differential equations when $F \equiv ||\nabla u||^2 + ||\nabla v||^2$.

(f) [3 marks] For a general F, suppose, in addition, that the vector field must also be divergence free on Ω (for example, if $(u, v)^T$ denotes an incompressible fluid flow in Ω): that is,

$$0 \equiv u_x(x, y) + v_y(x, y), \quad (x, y) \in \Omega.$$

Deduce a pair of partial differential equations that the extremal (u, v) must satisfy.

Again, give the resulting partial differential equations when $F \equiv ||\nabla u||^2 + ||\nabla v||^2$.

Further Partial Differential Equations

5. Consider a material that lies in $0 \leqslant x \leqslant 1$, which is liquid for $x < s(t)$ and solid for $x \geqslant s(t)$, where t denotes time. The dimensionless temperature is denoted by T_ℓ in the liquid and T_s in the solid and is governed by

$$\mathrm{St}\frac{\partial T_\ell}{\partial t} = \frac{\partial^2 T_\ell}{\partial x^2} + q, \qquad \text{for} \quad x < s(t), \qquad \mathrm{St}\frac{\partial T_s}{\partial t} = \frac{\partial^2 T_s}{\partial x^2} + q, \qquad \text{for} \quad x \geqslant s(t), \qquad \text{(1a,b)}$$

where $\mathrm{St} \ll 1$ denotes the Stefan number and $q > 2$ is a global heat source. These equations are accompanied by the following boundary conditions:

$$\frac{\partial T_\ell}{\partial x}(0,t) = 0, \qquad T_\ell(s(t),t) = 0, \qquad T_s(s(t),t) = 0, \qquad T_s(1,t) = -1, \qquad \text{(2a–d)}$$

$$\frac{\partial T_s(s(t),t)}{\partial x} - \frac{\partial T_\ell(s(t),t)}{\partial x} = \frac{\mathrm{d}s}{\mathrm{d}t}. \qquad \text{(2e)}$$

(a) [4 marks] Explain the physical significance of the five boundary conditions (2a–e).

(b) [5 marks] By considering the system in the limit $\mathrm{St} = 0$, find expressions for $T_\ell(x,t)$, $T_s(x,t)$ and an ordinary differential equation for $s(t)$.

(c) [3 marks] By considering $\partial T_s(s(t),t)/\partial x$, show that the solid will be superheated if $s < 1 - \sqrt{2/q}$, and hence explain why the model breaks down if $s < 1 - \sqrt{2/q}$.

(d) [3 marks] Now suppose the liquid region is replaced by a mushy region, with temperature $T_m = 0$ and liquid fraction $\theta \in [0,1]$, which is governed by

$$\frac{\partial \theta}{\partial t} = q. \qquad (3)$$

The solid region $x \geqslant s(t)$ is still governed by (1b) subject to the boundary conditions (2c,d) but (2e) is replaced by

$$\frac{\partial T_s(s(t),t)}{\partial x} = 0. \qquad (4)$$

Solve (1b) with $\mathrm{St} = 0$ subject to (2c,d) and (4) to find an expression for T_s and s and use this result to show that the solid is no longer superheated.

(e) [3 marks] Suppose initially the material is completely solid, so $s(0) = 0$ and $T_s(x,0) = -1$ for $0 \leqslant x \leqslant 1$. Explain why (3) indicates that pure liquid will first start to appear a time $1/q$ after the mushy layer first forms.

(f) [7 marks] Use the results of parts (a)–(e) and the fact that $\mathrm{St} \ll 1$ to explain how the initially entirely solid domain $T_s(x,0) = -1$ melts due to the global heat source, by (i) describing how the interface $s(t)$ moves, (ii) explaining which two phases the interface divides at each stage (*i.e.*, solid, liquid or mush), and (iii) stating the final configuration reached. (It is sufficient to write the interface motion in terms of differential equations and you do not need to solve these explicitly.)

6. Consider the following partial differential equation for the function $h(x,z,t)$:

$$\frac{\partial h}{\partial t} + h\frac{\partial h}{\partial x} + \gamma h^2 \frac{\partial h}{\partial z} = 0, \tag{1}$$

where $\gamma > 0$ is a constant.

(a) [5 marks] Suppose first that $h = h(x,t)$. Show that a similarity solution exists of the form $h = f(\eta)$ where $\eta = x/t^\alpha$ for some α that you should find. Find the ordinary differential equation satisfied by f and solve this to determine f.

(b) [5 marks] Now suppose that $h = h(x,z,t)$. By considering a scaling argument in which you balance terms in (1), or otherwise, show that a similarity variable of the form $\nu = zx^\beta t^\delta$ exists, for some β and δ that you should state.

(c) [5 marks] By seeking a solution of the form $h(x,z,t) = f(\eta)g(\nu)$, where f and η are the solution found in part (a) and ν is the similarity variable found in part (b), show that g satisfies the following ordinary differential equation:

$$g(g-1) + \nu(1-2g)g' + \gamma g^2 g' = 0. \tag{2}$$

(d) [5 marks] Suppose that $\gamma = 0$. Find the solution to (2) in this case. Show that, for a particular choice in the integration constant, the solution is consistent with the result found in part (a).

(e) [5 marks] Now suppose that $\gamma \ll 1$. Find a solution to (2) of the form $g(\nu) = 1 + \gamma g_1(\nu)$ for some function $g_1(\eta)$. Write down the solution for $h(x,z,t)$ in this case.