DEGREE OF MASTER OF SCIENCE

MATHEMATICAL MODELLING AND SCIENTIFIC COMPUTING

A2 Mathematical Methods II

TRINITY TERM 2020

THURSDAY, 14 May 2020
Opening Time: 9.30am (BST)
You have 3 hours 30 minutes to complete the paper and upload your answers.

This exam paper contains three sections.
You may attempt as many questions as you like but you must answer at least one question in each section. Your best answer in each section will count, along with your next best answer, making a total of four answers.

Please start the answer to each question on a fresh page.
All questions will carry equal marks.

Do not turn this page until you are told that you may do so

Section A: Nonlinear Systems

1. (a) [5 marks] Consider a dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ for $\mathbf{x} \in E \subseteq \mathbb{R}^n$, $\mathbf{f} \in C^r(E)$, with $r \geqslant 1$. Suppose that $\mathbf{x}_0$ is a fixed point. What does it mean for $\mathbf{x}_0$ to be *hyperbolic*? Under what conditions will this fixed point have a non-trivial (i) *local stable manifold*; (ii) *local unstable manifold*; (iii) *local centre manifold*? How are the stable and unstable manifolds defined?

   (b) [10 marks] Consider the system

$$\begin{aligned} \dot{x} &= -x - y + x^2, \\ \dot{y} &= -x^2 - xy. \end{aligned}$$

   Show that the origin is not a hyperbolic fixed point. Compute a cubic approximation to the local centre maniold. Find the local dynamics on this manifold, and hence determine the stability of the origin.

   (c) Consider the system

$$\begin{aligned} \dot{x} &= -2x - 3y + x^2, \\ \dot{y} &= 2x + y. \end{aligned}$$

   (i) [2 marks] Show that the origin is asymptotically stable.

   (ii) [2 marks] Show that the function $V(x, y) = x^2 + bxy + cy^2$ is positive for all $(x, y) \neq (0, 0)$ providing $4c > b^2$.

   (iii) [6 marks] Find constants b and c so that V is a Lyapunov function for the system above for sufficiently small x and y.

2. (a) [4 marks] Consider a dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ for $\mathbf{x} \in E \subseteq \mathbb{R}^n$, $\mathbf{f} \in C^r(E)$, with $r \geqslant 1$. What does it mean for a set $V \subseteq E$ to be an *invariant set*? When is an invariant set also an *attracting set*?

   (b) Consider the system

$$\begin{aligned} \dot{x} &= -x - 2y - x^2 - 4xy - 4y^2, \\ \dot{y} &= y + x^2 + 2xy + 2y^2. \end{aligned}$$

   (i) [3 marks] Show that the origin is a saddle point.

   (ii) [7 marks] Find a cubic approximation to the local stable and unstable manifolds.

   (iii) [3 marks] Show that the line $y = -x$ is an invariant set.

   (iv) [3 marks] Show that $J = x^3/3 + xy + x^2y + y^2 + 2xy^2 + 4y^3/3$ is a constant of the motion.

   (v) [5 marks] Sketch the level set $J = 0$. Use this to sketch the phase plane.

Section B: Further Mathematical Methods

3. (a) [8 marks] Given the integral equation,

$$y(x) = \lambda \mathcal{L} y(x) + f(x) = \lambda \int_a^b K(x,t) y(t) dt + f(x),$$

state the Fredholm Alternative Theorem for a general kernel K. What more can be said if K is symmetric ($K(x,t) = K(t,x)$)?

(b) [12 marks] Solve the inhomogeneous Fredholm integral equation for $y(x)$,

$$y(x) = 1 + x + \lambda \int_0^1 (1 + 5xt) y(t) dt,$$

noting carefully different cases for λ.

(c) [5 marks] Consider the homogeneous Fredholm equation,

$$y(x) = \lambda \int_0^1 e^{xt} y(t) dt.$$

Show how one can use a series approximation of the kernel to approximate the eigenvalues and eigenvectors to the full system. Give the general expression for each term of the resulting linear system (for a truncated series approximation). Do not evaluate large determinants or compute eigenvectors explicitly but derive the exact linear system from which they come, and describe how to compute the approximate solution for each approximate eigenvalue.

4. (a) Suppose the function $u(x)$ minimizes the functional,

$$J[u] = \int_a^b F(x, u, u') dx,$$

over all functions $u \in C^2([a,b])$ with $u(a) = c$ and $u(b) = d$, where F is continuously differentiable in all arguments.

(i) [10 marks] State the Fundamental Lemma of the Calculus of Variations, and use it to derive the Euler-Lagrange equations that u must satisfy.

(ii) [3 marks] Define the Hamiltonian H. What can be said about the Hamiltonian if F is autonomous (i.e. $F = F(u, u')$)?

(b) Consider a function $u(z) > 0$, and an associated surface $x^2 + y^2 = u(z)^2$ bounded between two planes $z = a$ and $z = b$.

(i) [3 marks] Explain briefly why the area of this surface lying between the two planes $z = a$ and $z = b$ is given by the functional,

$$J[u] = 2\pi \int_a^b u \sqrt{1 + (u')^2} dz.$$

(ii) [9 marks] Compute the solution u which minimizes this surface area between the circles $x^2 + y^2 = u^2$ with $u(-1) = \cosh(1)$ at $z = -1$ and $u(1) = \cosh(1)$ at $z = 1$. *Hint: You may find it useful to compute the expression* $\frac{d}{dz} \frac{u}{\sqrt{1+(u')^2}}$.

Section C: Further PDEs

5. Consider the one-dimensional gravity-driven spreading of a liquid that is poured onto a horizontal surface as governed by the dimensionless equation

$$\frac{\partial h}{\partial t} + \frac{\partial Q}{\partial x} = 0, \qquad \text{where} \quad Q = -h^3\frac{\partial h}{\partial x}. \tag{1}$$

Here, h denotes the height of the liquid, x the horizontal position and Q is the flux of liquid. Consider the case where we start with a clean surface so that $h(x,0) = 0$. Suppose that liquid is poured on at position $x = 0$ so that at time t the liquid covers a region of the surface defined by $-x_f(t) \leqslant x \leqslant x_f(t)$ where $x_f(t)$ is the position of the liquid front. This can be modelled by considering the region $0 \leqslant x \leqslant x_f(t)$ and imposing a non-zero flux Q at $x = 0$.

(a) [1 mark] Write down a condition on the flux Q at the front $x = x_f$.

(b) [5 marks] Suppose that the liquid is poured in such a way that the amount of liquid on the surface at time t satisfies

$$\int_0^{x_f(t)} h(x,t)\,\mathrm{d}x = t^\gamma, \tag{2}$$

where $\gamma > 0$ is a constant. By integrating (1) find a condition on the flux Q at $x = 0$ in terms of t and γ.

(c) [5 marks] Perform a scaling analysis on the governing equation (1) and the conservation law (2) to show that similarity solutions are of the form

$$h = t^{(2\gamma-1)/5} f(\eta), \qquad \text{where} \quad \eta = \frac{x}{t^{(3\gamma+1)/5}}.$$

(d) [4 marks] Show that the equation governing the height is

$$\frac{3\gamma+1}{5}\eta f' + \frac{1-2\gamma}{5} f + \left(f^3 f'\right)' = 0,$$

and the position of the front $\eta_f = x_f/t^{(3\gamma+1)/5}$ satisfies

$$\int_0^{\eta_f} f\,\mathrm{d}\eta = 1.$$

(e) [5 marks] By performing a local analysis near the front, show that

$$f^2 f' \sim -\frac{3\gamma+1}{5}\eta_f \qquad \text{as} \quad \eta \to \eta_f.$$

(f) [5 marks] Now suppose that the surface is initially covered with a liquid layer with initially uniform thickness $h(x,0) = 1$ and $h(x,t) \to 1$ as $x \to \pm\infty$. Using scaling arguments, show that any similarity solution as liquid is poured onto the surface is now of the form $h = g(x/t^{1/2})$ and find the equation satisfied by g.

6. Consider the melting of a one-dimensional material with a solid–liquid interface located at $x = s(t)$ with liquid and solid density ρ.

   (a) [3 marks] Write down the Stefan condition that relates the motion of this interface to the heat flux, $Q_i = -k_i \partial T_i/\partial x$, on the liquid ($i = \ell$) and solid ($i = s$) sides of the interface and the latent heat, L.

   (b) [5 marks] Consider the dimensionless two-phase Stefan problem of melting a solid by applying a heat source at $x = 0$:

$$
\begin{aligned}
&\mathrm{St}\frac{\partial T_\ell}{\partial t} = \frac{\partial^2 T_\ell}{\partial x^2} && 0 < x < s(t), \quad t > 0, && \text{(3a)}\\
&\frac{\mathrm{St}}{\kappa}\frac{\partial T_s}{\partial t} = \frac{\partial^2 T_s}{\partial x^2} && s(t) < x < 1, \quad t > 0, && \text{(3b)}\\
&T_\ell = 1 && x = 0, \quad t > 0, && \text{(3c)}\\
&\frac{\partial T_s}{\partial x} = 0 && x = 1, \quad t > 0, && \text{(3d)}\\
&T_\ell = T_s = 0, && x = s(t), \quad t > 0 && \text{(3e)}\\
&K\frac{\partial T_s}{\partial x} - \frac{\partial T_\ell}{\partial x} = \frac{\mathrm{d}s}{\mathrm{d}t} && x = s(t), \quad t > 0 && \text{(3f)}\\
&s = 0 && t = 0, && \text{(3g)}\\
&T_s = -\theta && t = 0, && \text{(3h)}
\end{aligned}
$$

   where St, κ, K and θ are all constants.

   Derive expressions for T_ℓ, T_s and s that satisfy the governing equations ($3a,b$) and the boundary conditions ($3c,d,e,f$) and initial condition ($3g$) in the limit $\mathrm{St} \to 0$. Does this solution satisfy the initial condition ($3h$)?

   (c) [4 marks] Now consider the early-time behaviour of the Stefan problem outlined in part (b). Show that the scalings

$$t = \mathrm{St}\,\tau, \quad x = \mathrm{St}^{1/2}X, \quad s(t) = \mathrm{St}^{1/2}S(\tau), \quad T_\ell(x,t) = \mathcal{T}_\ell(X,\tau), \quad T_s(x,t) = \mathcal{T}_s(x,\tau),$$

   give rise to the following leading-order problems in the limit $\mathrm{St} \to 0$:

$$\frac{\partial^2 \mathcal{T}_\ell}{\partial X^2} = 0 \qquad \mathcal{T}_\ell(0,\tau) = 1 \qquad \mathcal{T}_\ell\big(S(\tau),\tau\big) = 0, \qquad -\frac{\partial \mathcal{T}_\ell}{\partial X}\big(S(\tau),\tau\big) = \frac{\mathrm{d}S}{\mathrm{d}\tau}$$

   in $0 < X < S(\tau)$, and

$$\frac{\partial \mathcal{T}_s}{\partial \tau} = \kappa\frac{\partial^2 \mathcal{T}_s}{\partial x^2} \qquad \mathcal{T}_s(0,\tau) = 0, \qquad \frac{\partial \mathcal{T}_s}{\partial x}(1,\tau) = 0, \qquad \mathcal{T}_s(x,0) = -\theta$$

   in $0 < x < 1$.

   (d) [6 marks] Using the result of (c), obtain the leading-order solution for $\mathcal{T}_\ell$ and verify that it matches with the solution for T_ℓ obtained in (b) as $\tau \to \infty$.

   (e) [7 marks] Using the result of (c), obtain the leading-order solution for $\mathcal{T}_s$ using separation of variables or otherwise and verify that this also matches with the solution for T_s obtained in (b) as $\tau \to \infty$.