A11475W1

MASTER OF SCIENCE

Mathematical Modelling and Scientific Computing

Mathematical Methods II

TRINITY TERM 2024

Thursday 18 April 2024, 9:30am - 12:00pm

This exam paper contains three sections. You may attempt as many questions as you like but you must answer at least one question in each section. Your best answer in each section will count, along with your next best answer, making a total of four answers.

Do not turn this page until you are told that you may do so

Nonlinear Dynamics, Bifurcations and Chaos

1. Let matrix $M \in \mathbb{R}^{3\times 3}$ be given as

$$M = \begin{pmatrix} -3 & -1 & 4 \\ 1 & -1 & 0 \\ -1 & -1 & 2 \end{pmatrix}$$

   (a) [8 marks] Find the stable, unstable and center subspaces E^s, E^u and E^c of the linear system of ordinary differential equations

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = M\mathbf{x} \tag{$*$}$$

   where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

   (b) [9 marks] Consider the nonlinear system

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = M\mathbf{x} + \begin{pmatrix} 4\,x_1^2 \\ 0 \\ 0 \end{pmatrix}. \tag{$**$}$$

   Consider the fixed point at the origin $\mathbf{0} = [0,0,0]$. Find an approximation to the center manifold close to the fixed point $\mathbf{0}$ up to and including all quadratic terms, *i.e.* you are asked to express the center manifold locally as

$$c_1\,x_1 + c_2\,x_2 + c_3\,x_3 + c_4\,x_1^2 + c_5\,x_2^2 + c_6\,x_3^2 + c_7\,x_1\,x_2 + c_8\,x_1\,x_3 + c_9\,x_2\,x_3 = 0,$$

   where c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8 and c_9 are constants, which you should calculate.

   (c) [8 marks] Determine the stability of the fixed point at the origin $\mathbf{0} = [0,0,0]$ for:

   (i) the linear system $(*)$;

   (ii) the nonlinear system $(**)$.

2. (a) [8 marks] Let I be the interval

$$I = \left[-\frac{1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2} \right].$$

Let $x_0 \in I$. Define the sequence $x_k \in I$, $k = 0, 1, 2, \ldots$, iteratively by

$$x_{k+1} = 1 - x_k^2 .$$

(i) Find all fixed points of this map and determine their stability.

(ii) Find all 2-cycles of this map and determine their stability.

(b) [8 marks] Consider the following system of two ordinary differential equations

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}t} &= -\left(\frac{\mu+3}{2}\right) x_1 - \mu\, x_2 + \frac{x_1^3}{2} + 3\,x_1^2\,x_2 + 6\,x_1\,x_2^2 + 4\,x_2^3 \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= \mu\, x_1 + \left(2\mu - \frac{3}{2}\right) x_2 - x_1^3 - 6\,x_1^2\,x_2 - 12\,x_1\,x_2^2 - 8\,x_2^3 \end{aligned} \qquad (\bigstar)$$

where $\mu \in \mathbb{R}$ is a parameter.

(i) Linearize the system around the critical point at the origin $\mathbf{0} = [0, 0]$ in the form

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = M\mathbf{x}$$

where you should determine matrix $M \in \mathbb{R}^{2\times 2}$.

(ii) Show that there exists $\mu_c \in \mathbb{R}$ such that the origin $\mathbf{0}$ is asymptotically stable for $\mu < \mu_c$ and unstable for $\mu > \mu_c$.

(c) [9 marks] Consider the nonlinear system of ordinary differential equations given by ($\bigstar$). Find and classify all bifurcations as the parameter μ is varied, that is:

(i) Determine whether the bifurcation is a saddle-node bifurcation, transcritical bifurcation, supercritical pitchfork bifurcation or subcritical pitchfork bifurcation.

(ii) Determine the number and stability of critical points for the values of μ close to each bifurcation point.

Further Partial Differential Equations

3. Consider the partial differential equation

$$\frac{\partial u}{\partial t} = \frac{1}{x^n}\frac{\partial}{\partial x}\left(x^n u^m \frac{\partial u}{\partial x}\right), \tag{1}$$

where m and n are integers.

(a) [8 marks] Consider the case where $m = n = 0$ and $u \to 0$ as $x \to -\infty$ and $u \to 1$ as $x \to \infty$. Show that the partial differential equation (1) admits a similarity solution of the form $u = f(\eta)$ where $\eta = x/t^\alpha$ for some parameter α. Find the required value of α and the resulting ordinary differential equation satisfied by f. Solve this differential equation to determine the similarity solution $u(x,t)$.
Note: you may use the definition

$$\operatorname{erf}(\eta) = \frac{2}{\sqrt{\pi}}\int_0^\eta \mathrm{e}^{-s^2}\,\mathrm{d}s,$$

and the result $\operatorname{erf}(\infty) = 1$.

(b) Now suppose that $m = 2$ and $n = 1$.

(i) [3 marks] Introduce a new dependent variable

$$v = -u\frac{\partial u}{\partial x}.$$

Show that equation (1) may be written in terms of the following two first-order equations:

$$u\frac{\partial u}{\partial x} + v = 0, \qquad \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(uv) + \frac{uv}{x} = 0. \tag{2}$$

(ii) [3 marks] Given that the system (2) admits similarity solutions of the form

$$u = \frac{x}{\sqrt{t}}\,p(\xi), \qquad v = q(\xi), \qquad \xi = \frac{x}{t},$$

show that (2) may be transformed to the following simultaneous ordinary differential equations for p and q:

$$\xi p^2 + q + \xi^2 pp' = 0, \qquad 2pq - \xi^2 p' + \xi p' q + \xi p q' - \frac{1}{2}\xi p = 0, \tag{3}$$

where primes denote differentiation with respect to ξ.

(iii) [9 marks] By making the change of variables

$$p(\xi) = \sqrt{P(\xi)}, \qquad q(\xi) = \xi\, Q(\xi),$$

show that the system (3) may be transformed into two ordinary differential equations for P and Q and written in phase-plane form as

$$\frac{\mathrm{d}Q}{\mathrm{d}P} = F(P,Q), \tag{4}$$

for some function $F(P,Q)$ to be determined.

(iv) [2 marks] Explain how, through a change of independent variable $\xi = \xi(z)$, we may transform the ordinary differential equations obtained in part (iii) to a set of two autonomous ordinary differential equations.

4. (a) Consider the melting of a substance that is liquid for $-1 \leqslant x < s(t)$ and solid for $s(t) \leqslant x \leqslant 1$, where x denotes space, t denotes time, and $s(t)$ denotes the position of the solid–liquid interface. The temperature T is governed by the following system:

$$\frac{\partial^2 T}{\partial x^2} = 0, \qquad -1 \leqslant x \leqslant 1, \tag{1a}$$

$$T(-1,t) = 1, \qquad \frac{\partial T(1,t)}{\partial x} = 0, \tag{1b}$$

$$T = T_m, \qquad \left[\frac{\partial T}{\partial x}\right]_-^+ = \frac{\mathrm{d}s}{\mathrm{d}t}, \qquad \text{on } x = s(t), \tag{1c}$$

$$s = 0, \qquad \text{when } t = 0, \tag{1d}$$

where T_m is a constant.

(i) [5 marks] Explain the physics underpinning the system (1) and state any conditions on T_m for the problem to make physical sense.

(ii) [5 marks] Solve the system (1) for the temperature in both the liquid and the solid portions and for the position of the interface $s(t)$.

(b) Now suppose that there is an additive in the material with concentration $c(x,t)$, which satisfies

$$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2}, \qquad -1 \leqslant x \leqslant 1, \tag{2a}$$

$$\frac{\partial c(-1,t)}{\partial z} = 0, \qquad c(1,t) = c^*, \tag{2b}$$

$$[c]^- = -k_1 T_m, \qquad [c]^+ = -k_2 T_m, \tag{2c}$$

$$\left[\epsilon\frac{\partial c}{\partial x}\right]_-^+ = -\frac{\mathrm{d}s}{\mathrm{d}t}\,[c]_-^+\,, \tag{2d}$$

where $D = \epsilon \ll 1$ in $-1 \leqslant x < s(t)$ and $D = 0$ in $s(t) \leqslant x \leqslant 1$, and c^*, k_1 and k_2 are all positive constants.

(i) [4 marks] Explain the physics underpinning the system (2).

(ii) [6 marks] By making a change of variable $z = (x - s(t))/\epsilon$, explain why (2) reduces to the following system to leading order in ϵ:

$$\frac{\mathrm{d}s}{\mathrm{d}t}\frac{\partial c}{\partial z} + \frac{\partial^2 c}{\partial z^2} = 0, \quad z < 0, \tag{3a}$$

$$\frac{\partial c}{\partial z} \to 0 \quad \text{as } z \to -\infty, \tag{3b}$$

$$[c]^- = -k_1 T_m, \qquad \left[\frac{\partial c}{\partial z}\right]^- = \frac{\mathrm{d}s}{\mathrm{d}t}(k_1 - k_2)T_m \quad \text{on } z = 0, \tag{3c}$$

and $T = c^*$ for $z \geqslant 0$. You should also find a relationship between T_m and c^* and comment on the physical meaning of this expression.

(iii) [5 marks] Find the solution to (3) for the concentration $c(z,t)$ in $z < 0$ and show that this only exists if $\mathrm{d}s/\mathrm{d}t < 0$. Comment on the physical meaning for such a result and any potential complications that might arise in such a situation.

Further Mathematical Methods

5. Here we will use subscripts to denote differentiation of a function with respect to its arguments.

   (a) [3 marks] Let A denote an operator defined for suitable functions $u \in L_2[0,1]$, the Hilbert space of square integrable functions, $u(x)$, on [0,1], satisfying $u(0) = u(1) = 0$, where

   $$Au = u_{xx} + R^2 u$$

   for some positive constant R. Show that A is self-adjoint in $L_2[0,1]$.

   (b) [3 marks] Show that if $R = k\pi$ for $k = 1, 2, \dots$, then 0 is an eigenvalue of A with corresponding eigenfunction $\sin k\pi x$.

   (c) [3 marks] Consider the following nonlinear equation for $U(z)$ defined on the domain $[0, R]$:

   $$U_{zz} + U(1-U) = 0 \quad 0 < x < 1, \quad U(0) = U(R) = 0.$$

   Clearly $U = 0$ is a trivial solution for all $R > 0$.
   Let $u(x) = U(z)$, where $x = z/R$. Show that

   $$u_{xx} + R^2 u(1-u) = 0 \quad 0 < x < 1, \quad u(0) = u(1) = 0,$$

   so that

   $$Au - R^2 u^2 = 0.$$

   (d) [8 marks] Now consider the possible small solutions as R varies about π as follows: introduce a small parameter, ϵ, and set $R = \pi + \epsilon$.
   Expand any non-trivial equilibrium solution u as a regular asymptotic series in ϵ and use the Fredholm Alternative appropriately to show that

   $$u = \frac{3}{4}(R-\pi)\sin \pi x + O((R-\pi)^2).$$

   (e) [8 marks] Now for any integer $q \geqslant 2$, consider the small solutions, $u(x)$, of the equation:

   $$u_{xx} + R^2 u(1-u^q) = 0 \quad 0 < x < \pi, \quad u(0) = u(\pi) = 0.$$

   Again, introduce a small parameter ϵ, and set $R = \pi + \epsilon$. Expand u as a regular asymptotic series in ϵ^α, for an appropriate choice of $\alpha > 0$, and use the Fredholm Alternative appropriately to identify any small solutions, u, for R close to π.
   Comment on any differences between the cases where q is odd and even.

6. (a) [6 marks] Consider the Fredholm equation

$$u(x) = f(x) + \lambda \int_a^b K(x,t)\, u(t)\ \mathrm{d}t, \qquad x \in [a,b].$$

Here real λ, $f(x) \in L_2[a,b]$, and the kernel, $K(x,t)$, in $L_2[a,b]$ with respect to both of its arguments, are all given.

Write this equation as $Lu(x) = f(x)$ for an operator, L, defined on $L_2[a,b]$, and show directly that the adjoint operator, L^*, is given by

$$L^* v = v(x) - \lambda \int_a^b K(t,x)\, v(t)\ \mathrm{d}t, \qquad x \in [a,b],$$

for $v \in L_2[a,b]$.

(b) [6 marks] Now consider the case where K is a separable product of two non-zero functions, g and h:

$$K(x,t) = g(x)h(t).$$

Show that $L^*\tilde{y} = 0$, has a non-trivial solution if and only if it is in the form $\tilde{y}(x) = c\, h(x)$, for any constant c, and (λ, g, h) satisfy

$$1 = \lambda \int_a^b g(t)h(t)\, \mathrm{d}t.$$

Similarly show that $Ly = 0$ has a non-trivial solution if and only if it is in the form $y(x) = c\, g(x)$, for any constant c, and the same integral condition holds.

(c) [8 marks] Now consider the original equation, where again we have set $K(x,t) = g(x)h(t)$, so that we have:

$$y(x) - \lambda g(x) \int_a^b y(t)h(t)\ \mathrm{d}t = Ly(x) = f(x).$$

Take the inner product of this equation with $h(x)$ and show that

$$\left(\int_a^b y(t)h(t)\ \mathrm{d}t\right)\left(1 - \lambda \int_a^b g(x)h(x)\, \mathrm{d}x\right) = \int_a^b f(x)h(x)\, \mathrm{d}x.$$

Hence or otherwise show that :
(i) if $0 \neq \left(1 - \lambda \int_a^b g(x)h(x)\, \mathrm{d}x\right)$ a solution exists, and give its explicit form,
and
(ii) if $0 = \left(1 - \lambda \int_a^b g(x)h(x)\, \mathrm{d}x\right)$ then solutions exist if and only if $\int_a^b f(x)h(x)\, \mathrm{d}x = 0$, and give the general solution in that case.

(d) [5 marks] Let $[a,b] = [0,\pi]$ and $K(x,t) = \sin^2 x \cos^2 t$. For what critical value of λ must f satisfy a non-trivial condition in order for solutions to $Ly = f$ exist? State the condition on f in that case.

If f does not satisfy that condition and is fixed, what happens to the solution, y, as λ approaches that critical value?