1. (Solution) (a) [3 marks] f is a homeomorphism if it is a bijection, continuous and its inverse is continuous.

[2 marks] The closure of A is $\overline{A} = \bigcap\{B \subseteq M : A \subseteq B \text{ and } B \text{ closed}\}$.

(b) (i) [3 marks] Suppose for a contradiction that $f : \mathbb{R} \to \mathbb{R}^2$ is a homeomorphism. Then the induced map

$$\tilde{f} : \mathbb{R}\backslash\{0\} \to \mathbb{R}^2\backslash\{f(0)\}$$

is a homeomorphism. However the domain is disconnected and the codomain is connected. A contradiction and hence $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$.

(ii) [2 marks] This is clearly false. We might take $M = [0,1]$, $N = \mathbb{R}$ and take f to be the zero map. N is then not compact as it is not bounded.

(iii) [5 marks] This is true. Let $p, q \in f(M)$ and say that $f(a) = p$, $f(b) = q$. By the path-connectedness of M there is a continuous path $\gamma : [0,1] \to M$ such that $\gamma(0) = a$ and $\gamma(1) = b$. So $f \circ \gamma : [0,1] \to f(M)$ is a continuous path from p to q.

(iv) [5 marks] As f is a homeomorphism (so f and f^{-1} are continuous) then A is closed in M if and only if $f(A)$ is closed in N. Further, as f is a bijection, we have

$$\begin{aligned} f(\overline{A}) &= f\left(\bigcap\{B : A \subseteq B \text{ and } B \text{ closed in } M\}\right) \\ &= \bigcap\{f(B) : A \subseteq B \text{ and } B \text{ closed in } M\} \\ &= \bigcap\{f(B) : f(A) \subseteq f(B) \text{ and } f(B) \text{ closed in } N\} \\ &= \bigcap\{C : f(A) \subseteq C \text{ and } C \text{ closed in } N\} \\ &= \overline{f(A)}. \end{aligned}$$

(v) [5 marks] For any natural number N we have

$$\sum_{n=0}^{N}(x_n)^2 \leqslant \left(\sum_{n=0}^{N} |x_n|\right)^2$$

and so if $(x_n) \in l^1$ then $(x_n) \in l^2$. Also we have shown that

$$\|(x_n)\|_2 \leqslant \|(x_n)\|_1 .$$

Let $\varepsilon > 0$ and set $\delta = \varepsilon$. For $\|(x_n) - (y_n)\|_1 < \delta$ we then have $\|(x_n) - (y_n)\|_2 < \varepsilon$ and hence f is continuous.

2. (Solution) (a) (i) [6 marks]

- Note that $\delta \geqslant 0$ as $d_M \geqslant 0$ and $d_N \geqslant 0$. Further

$$\begin{aligned}\delta((m_1,n_1),(m_2,n_2)) = 0 \iff & \quad d_M(m_1,m_2) = 0 \quad \text{and} \quad d_N(n_1,n_2) = 0 \\ \iff & \quad m_1 = m_2 \quad \text{and} \quad n_1 = n_2 \\ \iff & \quad (m_1,n_1) = (m_2,n_2).\end{aligned}$$

- As d_M and d_N are symmetric then δ is clearly symmetric.
- Finally for (m_1,n_1), (m_2,n_2), (m_3,n_3) in $M \times N$ we can check two easy cases with

$$\begin{aligned}\delta((m_1,n_1),(m_2,n_2)) + \delta((m_2,n_2),(m_3,n_3)) &\geqslant d_M(m_1,m_2) + d_M(m_2,m_3) \geqslant d_M(m_1,m_3); \\ \delta((m_1,n_1),(m_2,n_2)) + \delta((m_2,n_2),(m_3,n_3)) &\geqslant d_N(n_1,n_2) + d_N(n_2,n_3) \geqslant d_N(n_1,n_3).\end{aligned}$$

  Hence $\delta((m_1,n_1),(m_2,n_2)) + \delta((m_2,n_2),(m_3,n_3)) \geqslant \delta((m_1,n_1),(m_3,n_3))$.

(ii) [4 marks] Let $\varepsilon > 0$. If $x_n \to x \in M$ then there exists N_1 such that $d_M(x_n,x) < \varepsilon$ for $n \geqslant N_1$. Likewise if $y_n \to y \in N$ then there exists N_2 such that $d_N(y_n,y) < \varepsilon$ for $n \geqslant N_2$. Hence

$$\delta((x_n,y_n),(x,y)) < \varepsilon \quad \text{for } n \geqslant \max\{N_1,N_2\}.$$

Conversely say that $\delta((x_n,y_n),(x,y)) \to 0$ as $n \to \infty$. By the sandwich rule $d_M(x_n,x) \to 0$ and $d_N(y_n,y) \to 0$ as $n \to \infty$.

(iii) [3 marks] Let (x_n,y_n) be a Cauchy sequence in $M \times N$. Let $\varepsilon > 0$. Then there exists K such that

$$\delta((x_n,y_n),(x_m,y_m)) < \varepsilon \quad \text{for } m,n \geqslant K.$$

Hence $d_M(x_n,x_m) < \varepsilon$ for $m,n \geqslant K$ and so (x_n) is Cauchy. By completeness $x_n \to x \in M$ and likewise $y_n \to y \in N$. By the previous part $\delta((x_n,y_n),(x,y)) \to 0$ and so $M \times N$ is complete.

(b) [6 marks] Consider the identity map $\iota : [0,1]^2 \to \left([0,1]^2, \delta\right)$ where the metric on the domain is the usual one, denoted d. We will show that it is continuous and that its inverse is continuous. Note for any real x, y we have

$$|x| \leqslant \sqrt{x^2+y^2} \leqslant 2\max\{|x|,|y|\}, \qquad |y| \leqslant \sqrt{x^2+y^2} \leqslant 2\max\{|x|,|y|\}$$

and so

$$\delta((x_1,y_1),(x_2,y_2)) \leqslant d((x_1,y_1),(x_2,y_2)) \leqslant 2\delta((x_1,y_1),(x_2,y_2)).$$

This shows that both ι and ι^{-1} are Lipschitz and so continuous.

(c) [2 marks] Now let $M = \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be the zero map which is clearly a contraction. Note that

$$F(x,y) = (y,0)$$

is not a contraction. For example consider the distance between $(0,0)$ and $(0,1)$ and their images.

[4 marks] On the other hand if $f : M \to M$ is a contraction with constant $K < 1$ then

$$F^2(m_1,m_2) = (f(m_1), f(m_2))$$

and we have

$$\begin{aligned}\delta(F^2(m_1,m_2),F^2(\mu_1,\mu_2)) &= \delta((f(m_1),f(m_2)),(f(\mu_1),f(\mu_2))) \\ &= \max\{d(f(m_1),f(\mu_1)), d(f(m_2),f(\mu_2))\} \\ &< K\max\{d(m_1,\mu_1), d(m_2,\mu_2)\} \\ &= K\delta((m_1,m_2),(\mu_1,\mu_2))\end{aligned}$$

and F^2 is a contraction as required.

3. (Solution)

(a) The Cauchy-Riemann equations state that $u_x = v_y$ and $u_y = -v_x$. [2 marks]

For $z = x + iy \in U$, we have

$$f'(z) = \lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \frac{u(x+h,y) + iv(x+h,y) - u(x,y) - iv(x,y)}{h} = u_x(x,y) + iv_x(x,y);$$

$$f'(z) = \lim_{\substack{ik\to 0\\ k\in\mathbb{R}}} \frac{u(x,y+k) + iv(x,y+k) - u(x,y) - iv(x,y)}{ik} = \frac{1}{i}\left(u_y(x,y) + iv_y(x,y)\right) = v_y(x,y) - iu_y(x,y).$$

Comparing real an imaginary parts, the result follows. [4 marks]

(b) (i) Each $z \in \mathbb{C}\backslash(-\infty, 0]$ can be written uniquely as $z = re^{i\theta}$ where $r > 0$ and $-\pi < \theta < \pi$.

We then define $L(z) = \log r + i\theta$. [3 marks]

We have

$$u = \log\sqrt{x^2+y^2} = \frac{1}{2}\log(x^2+y^2); \qquad v = \tan^{-1}\left(\frac{y}{x}\right).$$

Hence

$$u_x = \frac{x}{x^2+y^2}; \qquad v_y = \frac{1/x}{1+(y/x)^2} = \frac{x}{x^2+y^2};$$
$$u_y = \frac{y}{x^2+y^2}; \qquad v_x = \frac{-y/x^2}{1+(y/x)^2} = \frac{-y}{x^2+y^2}.$$

So the CREqs are met. [4 marks]

[1 mark] Also note

$$L'(z) = u_x + iv_x = \frac{x-iy}{x^2+y^2} = \frac{1}{z}.$$

(ii) [3 marks] We can see in $\mathbb{C}\backslash(-\infty, 0]$ that

$$\frac{\mathrm{d}}{\mathrm{d}z}(zL(z) - z) = L(z) + z \times \frac{1}{z} - 1 = L(z).$$

Hence

$$\int_{[1,i]} L(z)\,\mathrm{d}z = (iL(i) - i) - (1L(1) - 1) = i\left(\frac{\pi i}{2}\right) - i + 1 = 1 - \frac{\pi}{2} - i.$$

(iii) [2 marks] By definition the image $L(H) = \{z \in \mathbb{C} : 0 < \operatorname{Im} z < \pi\}$.

(c) (i) [3 marks] A conformal bijection of H which sends i to $\alpha \in H$ is

$$f_\alpha(z) = (\operatorname{Im}\alpha)z + \operatorname{Re}\alpha.$$

(ii) [3 marks] So a conformal bijection $R \to R$ which sends i to $\beta \in R$ is

$$g_\beta(z) = \frac{2}{\pi}(L \circ f_\alpha \circ \exp)(\pi z/2) \qquad \text{where} \quad \alpha = e^{\pi\beta/2}.$$

4. (Solution) (a) (i) [3 marks] *Morera's Theorem*: Let $f : U \to \mathbb{C}$ be a continuous function on a domain such that

$$\int_\gamma f(z)\,\mathrm{d}z = 0$$

for any closed path γ. Then f is holomorphic.

(ii) [5 marks] Note that f is continuous (and so integrable) as it is the uniform limit of continuous functions. Let $\varepsilon > 0$ and let $\mathcal{L}(\gamma)$ denote the length of γ. By uniform convergence there exists N such that

$$|f_n(z) - f(z)| < \frac{\varepsilon}{\mathcal{L}(\gamma)} \qquad \text{for } z \in \gamma \text{ and } n \geqslant N.$$

So by the Estimation Theorem, for $n \geqslant N$ we have

$$\left|\int_\gamma f_n(z)\,\mathrm{d}z - \int_\gamma f(z)\,\mathrm{d}z\right| = \left|\int_\gamma (f_n(z) - f(z))\,\mathrm{d}z\right| \leqslant \mathcal{L}(\gamma) \times \sup_{z\in\gamma} |f_n(z) - f(z)| \leqslant \mathcal{L}(\gamma) \times \frac{\varepsilon}{\mathcal{L}(\gamma)} = \varepsilon.$$

Hence $\int_\gamma f_n(z)\,\mathrm{d}z \to \int_\gamma f(z)\,\mathrm{d}z$ as $n \to \infty$.

(iii) [3 marks] Assume now that the above f_n are holomorphic, so f is still continuous, and assume now that γ is closed. By Cauchy's Theorem

$$\int_\gamma f_n(z)\,\mathrm{d}z = 0$$

for each n and so by part (ii)

$$\int_\gamma f(z)\,\mathrm{d}z = \lim_{n\to\infty} \int_\gamma f_n(z)\,\mathrm{d}z = 0.$$

By Morera's Theorem f is holomorphic.

(b) [5 marks] Let $a \in \mathbb{C}\backslash\mathbb{Z}$ and $r > 0$ such that $D(a,r) \subseteq \mathbb{C}\backslash\mathbb{Z}$. For $z \in D(a,r)$ and $n \in \mathbb{Z}$ we have

$$\left|n^2 - z^2\right| \geqslant n^2 - |z|^2 \geqslant n^2 - (|a| + r)^2.$$

Hence

$$\left|\frac{1}{n^2 - z^2}\right| \leqslant \frac{1}{n^2 - (|a| + r)^2} =: M_n.$$

As $\sum M_n$ converges then, by the Weierstrass M-Test, $\sum_{n=0}^\infty \frac{1}{n^2 - z^2}$ converges uniformly on $D(a,r)$ and so defines a holomorphic function on $D(a,r)$ and hence on $\mathbb{C}\backslash\mathbb{Z}$.

(c) Let Γ_n be as given and

$$\phi(w) = \frac{\pi}{(w^2 - z^2)\tan \pi w}.$$

Then ϕ has simple poles at $w \in \mathbb{Z}$ and $w = \pm z$. [2 marks] The residues at these poles are

$$\operatorname{res}(\phi; \pm z) = \frac{\pi}{\pm 2z \tan(\pm \pi z)} = \frac{\pi}{2z\tan(\pi z)} \qquad \text{[2 marks]}$$

$$\operatorname{res}(\phi; n) = \lim_{w\to n} \frac{\pi(w-n)}{(w^2 - z^2)\tan \pi w} = \frac{1}{n^2 - z^2} \lim_{w\to n} \frac{\pi}{\pi \sec^2 \pi w} = \frac{1}{n^2 - z^2}. \qquad \text{[2 marks]}$$

Hence for $N \geqslant |z|$ we have

$$\frac{1}{2\pi i}\int_{\Gamma_N} \phi(w)\,\mathrm{d}w = \left(\sum_{n=-N}^{N} \frac{1}{n^2 - z^2}\right) + \frac{\pi}{z \tan \pi z}.$$

However, by the Estimation Theorem,

$$\left|\int_{\Gamma_N} \phi(w)\,\mathrm{d}w\right| \leqslant \mathcal{L}(\Gamma_N) \times \frac{C}{N^2 - |z|^2} = O(N^{-1}) \to 0 \qquad \text{as } N \to \infty.$$

Hence, letting $N \to \infty$, we have

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 - z^2} = \frac{-\pi}{z \tan \pi z}. \qquad \text{[3 marks]}$$

5. (Solution) (a) [4 marks] By Laurent's Theorem, there exist unique $c_n \in \mathbb{C}$ such that

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n \qquad 0 < |z-a| < r.$$

We say that f has

- a removable singularity if $c_n = 0$ for all $n < 0$.
- a pole of order N if $c_{-N} \neq 0$ and $c_n = 0$ for $n < -N$.
- an essential singularity if $c_n \neq 0$ for infinitely many $n < 0$.

[1 mark] The residue of f at a is c_{-1}.

We have that $\sin z = z + O(z^3)$ and so by the Binomial Theorem

$$\frac{1}{\sin^2 z} = \frac{1}{(z+O(z^3))^2} = \frac{1}{z^2} \times \frac{1}{(1+O(z^2))^2} = \frac{1}{z^2}\left(1+O(z)^2\right)$$

is a pole of order 2. [2 marks] Again by the Binomial Theorem

$$\frac{1-\cos z}{1-\exp z} = \frac{1-(1-z^2/2+O(z^4))}{1-(1+z+O(z^2))} = \frac{z^2/2+O(z^4)}{-z+O(z^2)} = \frac{-z/2+O(z^3)}{1+O(z)} = -\frac{z}{2}+O(z^2)$$

showing the second singularity is removable. [3 marks]

(b) h is said to have a simple zero at a if $h(a) = 0 \neq h'(a)$. [1 mark] So we have by Taylor's Theorem

$$g(z) = g(a) + O(z-a), \qquad h(z) = h'(a)(z-a) + O((z-a)^2)$$

giving

$$\begin{aligned}
\frac{g(z)}{h(z)} &= \frac{g(a)+O(z-a)}{h'(a)(z-a)+O(z-a)^2} \\
&= \frac{1}{z-a} \times \frac{1}{h'(a)} \times \frac{g(a)+O(z-a)}{1+O(z-a)} \\
&= \frac{1}{z-a} \times \frac{1}{h'(a)} \times (g(a)+O(z-a)) \qquad \text{[by Binomial Theorem]} \\
&= \frac{g(a)/h'(a)}{z-a} + O(1). \qquad \text{[4 marks]}
\end{aligned}$$

(c) If we use the contour $\gamma(0,1)$, [1 mark] with the standard parametrisation $z = e^{i\theta}$, then we have

$$\begin{aligned}
\int_{\gamma(0,1)} z^{n-1} e^{-xz/2} e^{x/(2z)}\, dz &= \int_{\gamma(0,1)} z^{n-1} \exp\left(\frac{x(-z+z^{-1})}{2}\right) dz \\
&= \int_0^{2\pi} e^{(n-1)i\theta} \exp\left(\frac{-2ix\sin\theta}{2}\right) ie^{i\theta}\, d\theta \\
&= i\int_0^{2\pi} \exp\left(in\theta - ix\sin\theta\right) d\theta. \qquad \text{[3 marks]}
\end{aligned}$$

On the other hand, by Cauchy's Residue's Theorem [1 mark], noting the integrand has a singularity only at 0 [1 mark], we have

$$\begin{aligned}
\int_{\gamma(0,1)} z^{n-1} e^{-xz/2} e^{x/(2z)}\, dz &= 2\pi i \operatorname{res}\left(\sum_{k,l\geqslant 0} \frac{(-x)^k}{k!2^k} \frac{x^l}{l!2^l} z^{n-1+k-l}; 0\right) \\
&= 2\pi i \sum_{k\geqslant 0} \frac{(-x)^k}{k!2^k} \frac{x^{k+n}}{(k+n)!2^{k+n}} \\
&= 2\pi i \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!} \left(\frac{x}{2}\right)^{2n+k} \qquad \text{[3 marks]}
\end{aligned}$$

Taking imaginary parts [1 mark] we have

$$\int_0^{2\pi} \cos\left(n\theta - x\sin\theta\right) d\theta = 2\pi \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!} \left(\frac{x}{2}\right)^{2n+k}$$

and the result follows.

6. (Solution) (a) Let $S = \{(a,b,c) : a^2+b^2+c^2 = 1\}$ and $N = (0,0,1)$. Let π denote stereographic projection of S^2 from N to the extended complex plane $\widetilde{\mathbb{C}}$.

(i) [5 marks] The line connecting N to the point $P = (a,b,c) \in S$ is given parametrically as

$$\mathbf{r}(\lambda) = (0,0,1) + \lambda(a,b,c-1) = (\lambda a, \lambda b, 1+\lambda(c-1)).$$

This meats the plane $c = 0$ when $\lambda = 1/(1-c)$. We see that

$$\mathbf{r}\left(\frac{1}{1-c}\right) = \left(\frac{a}{1-c}, \frac{b}{1-c}, 0\right) \leftrightarrow \frac{a+ib}{1-c}.$$

[When $c = 1$, so that $P = N$, this is understood to be ∞ as intended.]

(ii) [5 marks] Let $f(z) = 1/z$ and $g = \pi^{-1} \circ f \circ \pi$. Note that

$$f(\pi(a,b,c)) = \frac{1-c}{a+ib} = \frac{(1-c)(a-ib)}{a^2+b^2} = \frac{(1-c)(a-ib)}{1-c^2} = \frac{a-ib}{1+c}.$$

It follows that $\pi(g(a,b,c)) = (a,-b,-c)$ (noting that $(a,-b,-c)$ is indeed on S).

(b) (i) [5 marks] The transformation

$$g(z) = \frac{(z_2-z_3)(z-z_1)}{(z_2-z_1)(z-z_3)}$$

clearly maps z_1, z_2, z_3 to $0, 1, \infty$ respectively and is a Möbius transformation as

$$\text{"}ad - bc\text{"} = \frac{(z_2-z_3)(z_1-z_3)}{(z_2-z_1)} \neq 0.$$

Further if h is a second such map then $h^{-1} \circ g$ is a Möbius transformation which fixes 0, 1, ∞. If

$$h^{-1}g(z) = \frac{az+b}{cz+d}$$

these respectively mean that $b = 0$, $c = 0$, $a = d$ so that $h^{-1}g = id$ and hence $g = h$.

(ii) [5 marks] Say for now that $z_1 = 1$, $z_2 = 2$, $z_3 = 3$. Let G denote the subgroup of Möbius transformations which fix the set $\{1,2,3\}$. If $\sigma \in S_3$ then there is a unique Möbius transformation f_σ such that

$$f_\sigma(i) = \sigma(i) \qquad \text{for } i = 1,2,3.$$

The map $S_3 \to G$ given by $\sigma \mapsto f_\sigma$ is a bijection (by part (i)) and is an isomorphism as both operations are composition.

If ϕ is the Möbius transformation which maps z_1, z_2, z_3 to $1, 2, 3$ then the desired subgroup is $\phi^{-1}G\phi \cong G \cong S_3$.

(iii) [5 marks] Arguing as in (ii) we may assume without any loss of generality that $z_1 = 0$ and $z_2 = \infty$. Let H denote the subgroup of Möbius transformations such that $0 \mapsto 0$ and $\infty \mapsto \infty$. If $f \in H$ and

$$f(z) = \frac{az+b}{cz+d}$$

then we need $b = 0$ to fix 0 and $c = 0$ to fix ∞. Hence

$$H = \left\{z \mapsto \frac{az}{d} : ad \neq 0\right\} = \{z \mapsto \lambda z : \lambda \neq 0\} \cong \mathbb{C}^*.$$