A2 Metric Spaces Model solutions and mark scheme

1. (a) (B) (3 marks) Let $C \subseteq \mathbb{R}^n$. Then C is compact if and only if C is closed and bounded.

(b) (B)

(i) (1 mark) Let C denote the compact metric space. Pick a point $p \in C$. For $i \in \mathbb{N}$, let U_i be the open ball around p of radius i. Note that the collection $\{U_i\}_{i \in \mathbb{N}}$ covers C.

(ii) (2 marks) If C is not bounded, then for every $r \in \mathbb{R}_+$, there are points $x, y \in C$ such that $d(x, y) > r$. By the triangle inequality, $d(x, p) + d(p, y) \geqslant d(x, y) > r$, so either $d(x, p)$ or $d(p, y)$ must be greater than $r/2$.

(iii) (1 mark) It follows that for every $i \in \mathbb{N}$, there is a point of C not in U_i. Thus there is no finite subcover of the given cover, and so C is not compact.

(c) (S)

(i) (1 mark) Not compact. The subspace is evidently not bounded.

(ii) (1 mark) Not compact. The subspace is not closed, as any irrational number between 0 and 1 is a limit of points in the subspace but is not itself in the subspace.

(iii) (1 mark) Compact. The subspace is closed and evidently bounded.

(iv) (1 mark) Not compact. The subspace is not closed, as for instance the sequence $\{(1 - 1/n, 0)\}$ converges in $\mathbb{R}^2$ but not in the subspace.

(v) (1 mark) Not compact. The subspace is not bounded, as the points $\{(n, 1/n) : n \in \mathbb{N}\}$ are contained in the subspace and have arbitrarily large norm.

(d) (N)

(i) (1 mark) By part (a), the subspace X must be either not bounded or not closed.

(ii) (4 marks) If X is not bounded, then the restriction to X of the continuous function $\mathbb{R} \to \mathbb{R}$, $x \mapsto e^{(x^2)}$ is continuous and not bounded.

(iii) (3 marks) If X is not closed, then there exists a point $p \in \mathbb{R}$ such that $p \notin X$ but there exists a sequence of points $x_i \in X$ converging to p; the restriction to X of the function $\mathbb{R} \backslash p \to \mathbb{R}$, $x \mapsto 1/(x - p)$ is continuous and not bounded.

(e) (N)

(i) (3 marks) If X is not bounded, then the restriction to X of the bounded continuous function $\mathbb{R} \to \mathbb{R}$, $x \mapsto (\arctan x)^2$ is continuous and bounded but does not obtain its bounds.

(ii) (2 marks) If X is not closed, then there exists a point $p \in \mathbb{R}$ such that $p \notin X$ but there exists a sequence of points $x_i \in X$ converging to p; the restriction to X of the function $\mathbb{R} \to \mathbb{R}$, $x \mapsto e^{(-(x-p)^2)}$ is continuous and bounded but does not obtain its bounds.

2. (a) (B)

(i) (2 marks) A metric space X is disconnected if there exist disjoint, nonempty, open subsets A and B of X such that $X = A \cup B$; a metric space is connected if it is not disconnected.

(ii) (2 marks) A metric space X is path-connected if for any two points a and b in X, there exists a continuous map $f : [0, 1] \to X$ such that $f(0) = a$ and $f(1) = b$.

(b) (B)

(i) (2 marks) Suppose $f : X \to \{a, b\}$ is a continuous map to the discrete set with two elements. By assumption, for any two points $x, y \in X$, there is a continuous map $\gamma : [0, 1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

(ii) (4 marks) The composite $f \circ \gamma : [0,1] \to \{a,b\}$ is continuous, therefore constant. Thus $f(x) = f(y)$, so f is the constant function and the inverse image of either $\{a\}$ or $\{b\}$ is empty. Therefore X is connected.

(c) (S)

(i) (1 mark) Not connected, not path connected. The given union is a partition into disjoint, nonempty, open subsets.

(ii) (1 mark) Connected, path connected. Given two points x, y, the straight line from x to the origin followed by the straight line from the origin to y is a path.

(iii) (1 mark) Connected, path connected. As in the previous part.

(iv) (1 mark) Connected, path connected. Given two points $(q,y), (q',y')$, the straight line from (q,y) to $(q,1)$ followed by the straight line from $(q,1)$ to $(q',1)$ followed by the straight line from $(q',1)$ to (q',y') is a path.

(v) (1 mark) Connected, path connected. Given two points (a,b), (c,d) of the space; without loss of generality assume a is rational. If c is rational, then the concatenation of the straight lines $(a,b)-(a,0)-(c,0)-(c,d)$ is a path; if d is rational, then the concatenation of the straight lines $(a,b)-(a,d)-(c,d)$ is a path.

(vi) (2 marks) Connected, not path connected.

(d) (N)

(i) (1 mark) Because S^2 is compact, there is a point $M \in S^2$ where the function f is maximal, and a point $m \in S^2$ where the function f is minimal.

(ii) (1 mark) Because S^2 is path-connected, we may pick a path $\gamma : [0,1] \to S^2$ from m to M.

(iii) (1 mark) Let $\alpha : S^2 \to S^2$ denote the antipodal function taking a point p to $-p$.

(iv) (1 mark) Consider the function $F := f \circ \gamma - f \circ \alpha \circ \gamma : [0,1] \to \mathbb{R}$.

(v) (1 mark) If $f(M) = f(-M)$ or $f(m) = f(-m)$, we are done.

(vi) (1 mark) Otherwise $f(-m) > f(m)$ so $F(0) < 0$, and $f(-M) < f(M)$ so $F(1) > 0$.

(vii) (1 mark) By the Intermediate Value Theorem, there exists a point $a \in [0,1]$ where $F(a) = 0$,

(viii) (1 mark) and so $f(\gamma(a)) = f(-\gamma(a))$ as required.

3. Solution: (a) [B] [4 marks] Let γ be a simple closed positively-oriented curve with a in its interior. Let f be holomorphic in and on γ. Then

$$f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a}\, \mathrm{d}z.$$

(b) [B] [10 marks] Let $P(z) = a_n z^n + a_{n-1} z^n + \cdots + a_0$ with $n \geqslant 1$ and $a_n \neq 0$. Then

$$\frac{P(z)}{z^n} \to a_n \qquad \text{as } z \to \infty.$$

So, taking $\varepsilon = |a_n|/2$ there exists $R > 0$ such that

$$\left| \frac{P(z)}{z^n} - a_n \right| < \frac{|a_n|}{2} \qquad \text{for } |z| > R,$$
$$\implies |P(z)| > \frac{1}{2} |a_n| |z|^n \qquad \text{for } |z| > R.$$

Suppose for a contradiction that P has no roots in $\mathbb{C}$ so that $1/P(z)$ is holomorphic. Then

$$\frac{2\pi i}{P(0)} = \int_{\gamma(0,r)} \frac{1}{zP(z)}\, \mathrm{d}z,$$

yet for any $r > R$ we have by the estimation theorem that

$$\left| \int_{\gamma(0,r)} \frac{1}{zP(z)}\, \mathrm{d}z \right| \leqslant 2\pi r \left(\frac{2}{|a_n| r^{n+1}} \right) = \frac{4\pi}{r^n} \to 0 \qquad \text{as } r \to \infty,$$

a contradiction.

(c) [S/N] [3 marks] We have $Q(z) = A(z - \alpha_1) \cdots (z - \alpha_n)$ for some A. By the product rule

$$Q(z) = A \sum_{i=1}^{n} \prod_{j=1, j\neq i}^{n} (z - \alpha_j)$$

so that

$$\frac{Q'(z)}{Q(z)} = \frac{1}{z - \alpha_1} + \cdots + \frac{1}{z - \alpha_n}.$$

[N] [4 marks] If $\operatorname{Im} \alpha_i > 0$ and $\operatorname{Im} \beta \leqslant 0$ then

$$\operatorname{Im} \left(\frac{1}{\beta - \alpha_i} \right) > 0 \qquad \text{for each } i$$

and hence $Q'(\beta)/Q(\beta) \neq 0$. In particular β is not a root.

[4 marks] By making an appropriate choice of coordinates (or performing a translation), we can now say that if a polynomial's roots are in a half-plane, then so are all its dertivative's roots. If each of the α_i lie in the disc $D(0, 1)$ and β lies outside the disc there is a half-plane that contains the α_i and not β, a contradiction.

4. Solution: (a) (i) [B] [2 marks]

$$c_n = \frac{1}{2\pi i}\int_{\gamma(0,r)} \frac{f(z)}{z^{n+1}}\,\mathrm{d}z.$$

(ii) [S] [5 marks] The z^n-term in $(1-z)^{-1/2}$ equals

$$\begin{aligned}
\frac{\frac{-1}{2}\times\frac{-3}{2}\times\cdots\times\frac{1-2n}{2}(-z)^n}{n!} &= \frac{1\times 3\times\cdots\times(2n-1)}{n!2^n}z^n \\
&= \frac{(2n)!}{(2\times 4\times\cdots\times 2n)n!2^n}z^n \\
&= \frac{(2n)!}{(2^n n!)n!2^n}z^n = \binom{2n}{n}\left(\frac{z}{4}\right)^n.
\end{aligned}$$

[S] [2 marks] Hence

$$\int_{\gamma(0,r)} \frac{\mathrm{d}z}{z^{n+1}\sqrt{1-z}} = \frac{2\pi i}{4^n}\binom{2n}{n}.$$

(b) [B] [3 marks] Let z be in the cut plane $\mathbb{C}\backslash[1,\infty)$ such that $1-z = re^{i\alpha}$ where $-\pi<\alpha<\pi$. Then

$$L(1-z) = \log r + i\alpha,$$

so that $L(1) = L(1e^{i0}) = \log 1 = 0$.

[S] [4 marks] If $0<\theta<2\pi$ then

$$\begin{aligned}
1-e^{i\theta} &= 1-\cos\theta - i\sin\theta \\
&= 2\sin^2\frac{\theta}{2} - 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2} \\
&= 2\sin\frac{\theta}{2}(-i)\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right) \\
&= 2\sin\frac{\theta}{2}e^{i(\theta/2-\pi/2)}.
\end{aligned}$$

[S] [2 marks] Hence, by definition,

$$\begin{aligned}
\sqrt{1-e^{i\theta}} &= \sqrt{2\sin(\theta/2)}e^{i\theta/4}e^{-i\pi/4} \\
&= \sqrt{2\sin(\theta/2)}e^{i\theta/4}\left(\frac{1-i}{\sqrt{2}}\right) \\
&= (1-i)\sqrt{\sin(\theta/2)}e^{i\theta/4}
\end{aligned}$$

[S/N] (c) [7 marks] Parametrising the $\gamma(0,1)$ integral with $z=e^{i\theta}$ we obtain

$$\int_{\gamma(0,1)} \frac{\mathrm{d}z}{z^{n+1}\sqrt{1-z}} = \int_0^{2\pi}\frac{ie^{i\theta}\mathrm{d}\theta}{e^{(n+1)i\theta}\sqrt{1-e^{i\theta}}} = \int_0^{2\pi}\frac{ie^{-in\theta}\mathrm{d}\theta}{(1-i)\sqrt{\sin(\theta/2)}e^{i\theta/4}} = \int_0^{2\pi}\frac{ie^{-i(n+1/4)\theta}\mathrm{d}\theta}{(1-i)\sqrt{\sin(\theta/2)}}.$$

Using the given equality, we know that

$$\int_0^{2\pi}\frac{e^{-i(n+1/4)\theta}\mathrm{d}\theta}{\sqrt{\sin(\theta/2)}} = \frac{2\pi(1-i)}{4^n}\binom{2n}{n}.$$

Taking real parts we have

$$\int_0^{2\pi}\frac{\cos(n+1/4)\theta}{\sqrt{\sin(\theta/2)}}\mathrm{d}\theta = \frac{2\pi}{4^n}\binom{2n}{n}.$$

Substituting $\theta = 2t$ we get

$$\int_0^{\pi}\frac{\cos(2n+1/2)t}{\sqrt{\sin t}}\mathrm{d}t = \frac{\pi}{4^n}\binom{2n}{n},$$

as required.

5. Solution: (a) [B] [4 marks] Let γ be a simple, closed positively-oriented curve. Let f be a function which is holomorphic in and on γ except for finitely many singularlities $a_1, a_2, \ldots, a_n$ inside γ. Then

$$\int_\gamma f(z)\,\mathrm{d}z = 2\pi i \sum_{k=1}^{n} \operatorname{res}(f; a_k).$$

(b) [S] (i) [4 marks] Let $x, y \in \mathbb{R}$. Then

$$\begin{aligned}
\sinh x \cos y + i \cosh x \sin y &= \left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^{iy} + e^{-iy}}{2}\right) + i\left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^{iy} - e^{-iy}}{2i}\right) \\
&= \frac{1}{4}\left\{\left(e^{x+iy} + e^{x-iy} - e^{-x+iy} - e^{-x-iy}\right) + \left(e^{x+iy} - e^{x-iy} + e^{-x+iy} - e^{-x-iy}\right)\right\} \\
&= \frac{1}{4}\left\{2e^{x+iy} - 2e^{-x-iy}\right\} = \sinh(x+iy).
\end{aligned}$$

(ii) [3 marks] By the triangle inequality

$$|\sinh(x+iy)| = \frac{1}{2}\left|e^{x+iy} - e^{-x-iy}\right| \geqslant \frac{1}{2}\left|\left|e^{x+iy}\right| - \left|e^{-x-iy}\right|\right| = \frac{1}{2}\left|e^x - e^{-x}\right| = |\sinh x|.$$

(iii) [5 marks] Note

$$\begin{aligned}
\sinh^2(x+iy) = -\cosh^2 a \implies \quad & \sinh(x+iy) = \pm(\cosh a)\, i \\
\implies \quad & \sinh x \cos y = 0 \quad \text{and} \quad \cosh x \sin y = \pm(\cosh a).
\end{aligned}$$

If $\sinh x = 0$ then $\cosh x = 1$ and $|\sin y| = \cosh a > 1$, a contradiction. So $\cos y = 0$ and $\sin y = \pm 1$. Also note $\cosh x = -\cosh a$ is impossible. Hence one of the following holds:

$$\begin{aligned}
\cos y &= 0, \quad \sin y = 1, \quad \cosh x = \cosh a; \\
\cos y &= 0, \quad \sin y = -1, \quad \cosh x = \cosh a;
\end{aligned}$$

so that the solutions are

$$z_n^{\pm} = \pm a + (2n+1)\,\pi i/2.$$

(c) [S/N] [1 mark] The only singularities in the suggested contour are

$$\alpha = a + \pi i/2 \qquad \text{and} \qquad \beta = -a + \pi i/2.$$

We have (from the given hint)

$$\sinh\alpha = (\cosh a)i, \qquad \sinh\beta = (\cosh a)i, \qquad \cosh\alpha = (\sinh a)i \qquad \cosh\beta = -(\sinh a)i.$$

Let $g(z) = \sinh^2 z + \cosh^2 a$. As

$$g'(\alpha) = 2\sinh\alpha\cosh\alpha = -2\sinh a\cosh a = -\sinh 2a, \qquad g'(\beta) = \sinh 2a$$

are both non-zero then α and β are simple poles [2 marks] with residues [2 marks]

$$\operatorname{res}(f, \alpha) = \frac{-\alpha}{\sinh 2a}, \qquad \operatorname{res}(f, \beta) = \frac{\beta}{\sinh 2a}.$$

By Cauchy's Residue Theorem

$$\int_{\Gamma_R} f(z)\,\mathrm{d}z = \frac{2\pi i}{\sinh 2a}\{\beta - \alpha\} = \frac{-4\pi i a}{\sinh 2a}. \qquad \text{[1 mark]}$$

Also by (b)(i) the contributions from the left and right edges satisfy

$$\left|\int_{\text{edge}} f(z)\,\mathrm{d}z\right| \leqslant \pi\frac{(R+\pi)}{\sinh R} \to 0 \text{ as } R \to \infty. \qquad \text{[2 marks]}$$

Letting $R \to \infty$ we have

$$\int_{-\infty}^{\infty} \frac{x\,\mathrm{d}x}{\sinh^2 x + \cosh^2 a} - \int_{-\infty}^{\infty} \frac{(x+\pi i)\,\mathrm{d}x}{\sinh^2(x+\pi i) + \cosh^2 a} = \frac{-4\pi i a}{\sinh 2a}.$$

As $\sinh(x + \pi i) = -\sinh x$ [1 mark] the result follows.

6. Solution: (a) [B] [8 Marks] If $A \neq 0$ then we can rewrite the given equation as

$$z\bar{z} + \frac{\overline{B}}{A}z + \frac{B}{A}\bar{z} + \frac{C}{A} = 0,$$

and completing the square, rewrite it further as

$$\left|z + \frac{B}{A}\right|^2 = \frac{|B|^2 - AC}{A^2},$$

This represents a circle if $|B|^2 \geqslant AC$ and is empty otherwise. If $A = 0$ then the equation is

$$\mathrm{Re}(\overline{B}z) = -2C$$

which represents a line, as a dilation enlarges and rotates the plane.

(b) [B/S] [9 marks] The circline

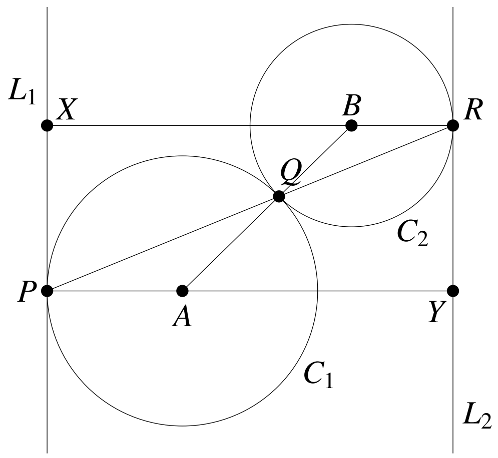
$$Az\bar{z} + \overline{B}z + B\bar{z} + C = 0,$$

maps to

$$A\left(\frac{1}{z}\right)\left(\frac{1}{\bar{z}}\right) + \overline{B}\frac{1}{z} + B\frac{1}{\bar{z}} + C = 0 \implies Cz\bar{z} + \overline{B}\bar{z} + Bz + A = 0.$$

- For a circle through the origin, $A \neq 0 = C$ and so the image is a line not through the origin.
- For a circle not through the origin $A \neq 0 \neq C$ and so the image is a to a circle not through the origin.
- For a line through the origin $A = 0 = C$ and so the image is a line through the origin.

(c) [N] [8 marks] If we draw in the horizontals through the circles centres (A and B), and the line segment between those centres, we have the following diagram.



[4 marks] AQB is a line as C_1 and C_2 are tangential. So $\angle QAY = \angle QBX$. But $\angle QBX = 2\angle QRB$ and $\angle QAY = 2\angle QPA$ by a circle theorem. So $\angle QPA = \angle QRB$ which shows that PQR is a line.

[4 marks] If we take the tangency of C_3 as C_4 as the origin and perform the map $1/z$ then C_3 and C_4 transform to two parallel lines L_1 and L_2 (as their tangency is now at ∞). The three remaining tangencies map to P, Q, R which are collinear with ∞. By part (b) their preimages are concyclic.