A2 Exam Paper 2019 - Model Solutions (Version: 1 Dec. 2018)

Q1. (a) [New, 8 marks=3+3+2]
Solution. (a,i) [3 marks] Since $\varphi \geq 0$, so $\rho(x,y) = \varphi(d(x,y)) \geq 0$. Since $d(x,y) = d(y,x)$

$$\rho(x,y) = \varphi(d(x,y)) = \varphi(d(y,x)) = \rho(y,x).$$

By assumption, $\varphi(t) = 0$ only when $t = 0$, hence $\rho(x,y) = \varphi(d(x,y)) = 0$ implies that $d(x,y) = 0$, so that $x = y$ as d is a distance. Finally verify that ρ also satisfies the triangle inequality. If x, y and z in X, we have $d(x,z) \leq d(x,y) + d(y,z)$, so that, by using the assumptions on φ we have

$$\begin{aligned}\rho(x,z) &= \varphi(d(x,z)) \leq \varphi(d(x,y) + d(y,z)) \leq \varphi(d(x,y)) + \varphi(d(y,z)) \\ &= \rho(x,y) + \rho(y,z).\end{aligned}$$

Therefore ρ is a metric on X.

(a, ii) [3 marks] Suppose $x_n \to x$ in (X,d), i.e. $d(x_n,x) \to 0$, since φ is (right) continuous at 0, $\varphi(d(x_n,x)) \to 0$, that is, $\rho(x_n,x) \to 0$.

The converse is also true, we may argue by contradiction. Suppose $\rho(x_n,x) \to 0$ but $d(x_n,x)$ does not tend to zero, thus there is $\varepsilon > 0$ and there is a sub-sequence $\{x_{n_k}\}$ such that $d(x_{n_k},x) \geq \varepsilon$, which implies that $\rho(x_{n_k},x) \geq \varphi(\varepsilon) > 0$ for all k, which is a contradiction.

(a,iii) [2 Mark] We may choose $\varphi(t) = \frac{t}{1+t}$ and $\rho(x,y) = \varphi(d(x,y))$. Then φ is increasing on $[0,\infty)$ and $\varphi(t) = 0$ only for $t = 0$. φ is continuous at 0, and

$$\frac{t+s}{1+t+s} = \frac{t}{1+t+s} + \frac{s}{1+s+t} \leq \frac{t}{1+t} + \frac{s}{1+s}$$

for all $s,t \geq 0$, that is, $\varphi(s+t) \leq \varphi(s) + \varphi(t)$. Therefore, by a) and b) , $d' = \rho$ is a metric equivalent to d. While $d'(x,y) \leq 1$ for all $x,y \in X$.

(b) [Similar, 5 marks =3+1+1] *Solution.* Since $A \subseteq \overline{A \cup B}$ so $\overline{A} \subseteq \overline{A \cup B}$ and similarly $\overline{B} \subseteq \overline{A \cup B}$. On the other hand $A \cup B \subseteq \overline{A} \cup \overline{B}$ and $\overline{A} \cup \overline{B}$ is closed, we therefore have $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$. Hence $\bar{A} \cup \overline{B} = \overline{A \cup B}$. [3 marks]

In general $\overline{A \cap B} = \overline{A} \cap \overline{B}$ is false, for example $A = \mathbb{Q}$ and $B = \mathbb{R} \setminus \mathbb{Q}$. Then $A \cap B = \emptyset$ but $\overline{A} = \overline{B} = \mathbb{R}$. [1 mark]

Moreover $A \cap \overline{B} = \mathbb{Q}$, $\overline{A} \cap B = \mathbb{R} \setminus \mathbb{Q}$, $\overline{A \cap B} = \emptyset$ and $\overline{A} \cap \overline{B} = \mathbb{R}$ are different. [1 mark]

(c) *[*Book, 7 marks=3 +4]
Solution. (c, i) [3 marks] $f : X \to X$ is a contraction if there is a constant $0 \leq c < 1$ such that $d(f(x), f(y)) \leq c d(x,y)$ for every $x, y \in X$. The Contraction Mapping Theorem says that if $f : X \to X$

is a contraction on a complete metric space (X,d), then f has a unique point, i.e. there is a unique $x \in X$, such that $f(x) = x$.

(c, ii) [4 marks] Let us consider the function $F(x) = d(g(x), x)$ for $x \in X$. Then

$$\begin{aligned} |F(x) - F(y)| &= |d(g(x),x) - d(g(y),y)| \\ &= |d(g(x),x) - d(x,g(y)) + d(x,g(y)) - d(g(y),y)| \\ &\leq d(g(x),g(y)) + d(x,y) \\ &\leq 2d(x,y) \end{aligned}$$

for all x, y, hence F is continuous on X. Since X is compact, F achieves its minimum on X, that is, there is $x_0 \in X$, such that

$$d(g(x_0), x_0) = \inf_{x \in X} d(g(x), x).$$

If $g(x_0) \neq x_0$, then $d(g^2(x_0), g(x_0)) < F(x_0)$ by assumption, so that $F(g(x_0)) < F(x_0)$ which contradicts to the claim x_0 is the inf of F. Thus x_0 must be a fixed point of g.

(d) [New, 5 marks 4 +1] *Solution.* Suppose $a \in B$, then $F(a) = \xi = (\xi_n)$ given in the question possesses the following properties:

$$\xi_n = a_{n-1}\left(1 - \frac{1}{2^n}\right) \to 0 \qquad \text{as } n \to \infty$$

and

$$|\xi_0| = \frac{1}{2}(1 + \|a\|) < 1.$$

While for $n \geq 1$ we have

$$|\xi_n| = |a_{n-1}|\left(1 - \frac{1}{2^n}\right) \leq |a_{n-1}| \leq \|a\|$$

so by definition $\|\xi\| \leq 1$, and therefore $F(a) \in B$ for every $a \in B$.

Suppose $a = (a_n)$, $b = (b_n) \in B$, and let $\xi = F(a)$ and $F(b) = \eta$. Then $\xi_0 - \eta_0 = \frac{1}{2}(\|a\| - \|b\|)$ and for $n \geq 1$

$$\xi_n - \eta_n = (a_{n-1} - b_{n-1})\left(1 - \frac{1}{2^n}\right)$$

so that

$$|\xi_0 - \eta_0| \leq \frac{1}{2}\|a - b\|$$

and for $n \geq 1$

$$|\xi_n - \eta_n| \leq \left(1 - \frac{1}{2^n}\right)|a_{n-1} - b_{n-1}|.$$

If $a \neq b$, then $\|a - b\| > 0$. Since $a_n - b_n \to 0$ so there is $N \in \mathbb{N}, |a_{n-1} - b_{n-1}| \leq \frac{1}{2}\|a - b\|$ for $n \geq N$. Hence

$$|\xi_n - \eta_n| \leq \left[\left(1 - \frac{1}{2^N}\right) \vee \frac{1}{2}\right]\|a - b\|$$

for all n, it follows that

$$\|F(a) - F(b)\| \leq \left[\left(1 - \frac{1}{2^N}\right) \vee \frac{1}{2}\right]\|a - b\| < \|a - b\|$$

as long as $a \neq b$.

We are going to show that F has no fixed point. Argue by contradiction. Suppose $a = F(a)$ where $a = (a_n) \in B$. Then, by definition of F

$$a_0 = \frac{1}{2}(1 + \|a\|)$$

and

$$a_n = a_{n-1}\left(1 - \frac{1}{2^n}\right)$$

for $n \geq 1$. It follows that

$$a_n = \prod_{k=1}^{n}\left(1 - \frac{1}{2^k}\right) a_0 = \frac{1}{2}(1 + \|a\|)\prod_{k=1}^{n}\left(1 - \frac{1}{2^k}\right)$$

which does not converge to zero, a contradiction to the assumption that $a \in B$. [4 marks]

By part (c), one can conclude that B is not compact. [1 marks] [An argument by using unit vectors is also fine].

Q2. (a) [Book, 3 marks] *Solution.* X is connected if $X = U \cup V$ where U, V are open and disjoint, then U or V is empty.

X is path-connected, if for every $x, y \in X$ there is a continuous mapping γ from $[0,1]$ to X such that $\gamma(0) = x$ and $\gamma(1) = y$.

(b) [Book+Similar, 10 marks=3+4+3] *Solution.*

(b, i) [3 marks] We begin with the proof of the first claim. Suppose $f : X \to \{0,1\}$ is continuous. Let $U = f^{-1}(0)$ and $V = f^{-1}(1)$. Then U, V are open (also closed), disjoint and $X = U \cup V$. If X is connected, the U or V is empty, so $f(x) = 0$ for all x, or $f(x) = 1$ for all x, accordingly. Conversely, if X is disconnected, so that $X = U \cup V$ where U, V are open disjoint and both are nonempty. Define $f(x) = 0$ for $x \in U$, and $f(x) = 1$ for $x \in V$. Then $f^{-1}(A) = \emptyset, X, U$ or V, hence f is continuous, and f is not constant.

(b, ii) [4 marks] Let us prove that $X \subseteq \mathbb{R}$ is connected if and only if $X = J$ is an interval. First prove any interval is connected. If X is an interval with end points $a \leq b$. ~~If $a = b$ then $X = [a,a] = \{a\}$ which is connected. Otherwise we~~ argue by contradiction. Suppose $f : X \to \{0,1\}$ is continuous, so f is also a continuous function from X to $[0,1]$. If f is not constant, then there are $x, y \in X$ such that $f(x) = 0$ and $f(y) = 1$. We may assume that $x < y$. Since X is an interval, so that $[x,y] \subseteq X$ and f is continuous on $[x,y]$. By IVT, there is $z \in [x,y]$ such that $f(z) = 1/2$, which is a contradiction.

Conversely, assume that X is connected, we show that X is an interval, that is, if $x < y$ and $x, y \in X$, we need to show that $(x,y) \subseteq X$. Argue by contradiction. Suppose $c \in (x,y)$ but $c \notin X$. Let $U = X \cap (-\infty, c)$ and $V = X \cap (c, \infty)$. Then U, V are open and disjoint, $x \in U$, $y \in V$, and $X = U \cup V$ as $c \notin X$. Thus X is disconnected, a contradiction.

(b, iii) [3 marks] Let us now prove the last statement. Suppose X is path-connected, and we show that X is connected. Again argue by contradiction. Suppose X is disconnected, that is $X = U \cup V$ where U, V are open, disjoint and both non-empty. Let $x \in U$, and $y \in V$. Since X is path-connected, so there is a continuous map $p : [0,1] \to X$ such that $p(0) = x$ and $p(1) = y$. Then

$$[0,1] = p^{-1}(U) \cup p^{-1}(V)$$

where $p^{-1}(U)$, $p^{-1}(V)$ are open, disjoint, and no one is empty, which would yield that the interval $[0,1]$ is disconnected, a contradiction.

(c) [Similar, 6 marks] *Solution.* Suppose $X \subseteq \mathbb{R}^n$ is open and connected. Let $a \in X$ be any but fixed point, and let $A = \{x \in X : x \text{ can be connected to } a\}$, i.e. A consists of all points $x \in X$, such that there is a continuous p from $[0,1]$ to X such that $p(0) = x$ and $p(1) = a$. We show that both A and A^c are both open. Suppose $x \in A$, then also $x \in X$. Since X is open in $\mathbb{R}^n$, so there is $r > 0$, the open ball (with center x and radius r) $B(x,r) \subseteq X$. Since any point $y \in B(x,r)$ can be connected to the center x then to a, so that $B(x,r) \subseteq A$. Therefore A is open. Similar argument shows that if A^c were non-empty, then A^c is open too. Since X is connected, and A is non-empty, so that $A = X$ which implies that X is path-connected.

(d) [New, 6 marks] *Solution.* Suppose there is 1-1 onto continuous $f : [0,1] \to S^1$, then f maps $[0,c) \cup (c,1]$ (which is disconnected by part 2)) one to one and onto $S^1 \setminus \{a\}$ where $a = f(c)$, and $c \in (0,1)$. Since f and f^{-1} are continuous, and f^{-1} maps a connected space $S^1 \setminus \{a\}$ to a disconnected one, which produces a contradiction.

Choose $a_n = \frac{1}{n}$ $(n = 1,2,\cdots)$. Let $f(x) = e^{i2\pi x}$ for $x \in [0,1)$ but $x \neq a_n$ for $n \geq 1$ and $f(a_n) = e^{i2\pi a_{n+1}}$ for $n \geq 1$.

Q3. (a) [Book, 6 marks=2+2+2] *Solution.* (a, i) [2 marks] f is holomorphic in an open subset $D \subseteq \mathbb{C}$, if for every $z \in D$, the complex derivative

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists for every $z \in D$. [*Equivalent definition is acceptable*].

(a,ii) [2 marks] By letting h = real but go to zero, we obtain that, if $f = u + vi$ is holomorphic,

$$f'(z) = u_x + v_x i$$

and by letting $h = i\delta$, where δ real and $\delta \to 0$, to obtain

$$f'(z) = \frac{1}{i}(u_y + v_y i)$$

and therefore

$$u_x + v_x i = \frac{1}{i}(u_y + v_y i)$$

which is equivalent to the Cauchy-Riemann equations:

$$u_x = v_y \text{ and } u_y = -v_x.$$

(a, iii) [2 marks] If $f = u + vi$ is real, so that $v = 0$, which implies $u_x = u_y = 0$ too by Cauchy-Riemann equations. Hence u is constant as D is connected and open. It follows that f is constant on D.

(b) [Book, 6 marks=3 +3] *Solution.* (b, i) [3 marks] Talyor's expansion: If f is holomrphic in D, and suppose $r > 0$ such that the disk $D(a,r) \subseteq D$, then

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n \text{ for any } z \in D(a,r)$$

where

$$a_n = \frac{1}{n!} f^{(n)}(a) = \frac{1}{2\pi i} \int_{C(a,\rho)} \frac{f(w)}{(w-a)^{n+1}} dw$$

where $C(a,\rho)$ is the circle with center a and radius ρ, as long as $0 < \rho < r$.

(b, ii) [3 marks] Suppose f is bounded and holomorphic in $\mathbb{C}$, then

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ for any } z$$

where

$$a_n = \frac{1}{2\pi i} \int_{C(0,R)} \frac{f(w)}{w^{n+1}} dw.$$

Suppose $|f(z)| \le M$ for any $z \in \mathbb{C}$, where M is a bound of f in $\mathbb{C}$. Then, by the Estimation Lemma,

$$\begin{aligned} |a_n| &\le \frac{1}{2\pi} 2\pi R \max_{|w|=R} \left| \frac{f(w)}{w^{n+1}} \right| = R \frac{1}{R^{n+1}} \max_{|w|=R} |f(w)| \\ &\le R \frac{1}{R^{n+1}} M \end{aligned}$$

for any $R > 0$. Therefore, for $n \ge 1$, by letting $R \to \infty$ we conclude that $|a_n| \le 0$, so $a_n = 0$ for all $n \ge 1$, hence $f(z) = a_0$ is constant.

(c) [Book-Similar-New, 13 marks = 2+2+4+5] *Solution.* (c, i) [*Book work*, 2 marks] It is said that a is an isolated singularity of f, if there is $r > 0$, f is holomorphic on $D(a,r) \setminus \{a\}$, where $D(a,r) \setminus \{a\} = \{z : 0 < |z-a| < r\}$. An isolated singularity is removable if there is a holomorphic function g in $D(a,r)$ (for some $r > 0$) such that $f = g$ on $D(a,r) \setminus \{a\}$, that is, f can be extended (uniquely) to a holomorphic function in $D(a,r)$.

(c, ii) [Book, 2 marks] Laurent's expansion. If b is an isolated singularity, so that there is $r > 0$ such that f is holomorphic in $D(b,r) \setminus \{b\}$, and

$$f(z) = \sum_{n=1}^{\infty} c_{-n}(z-b)^{-n} + \sum_{n=0}^{\infty} c_n(z-b)^n = \sum_{n=-\infty}^{\infty} c_n(z-b)^n \text{ for any } 0 < |z-b| < r$$

where

$$c_n = \frac{1}{2\pi i} \int_{C(b,\rho)} \frac{f(w)}{(w-b)^{n+1}} dw$$

for all $n \in \mathbb{Z}$, where $0 < \rho < r$.

(c, iii) [Similar, 4 marks] Suppose f is bounded on $D(b,r) \setminus \{b\}$, say $|f(z)| \le M$ for any $0 < |z-a| < r$. Then, by Estimation Lemma,

$$\begin{aligned} |c_n| &\le \frac{1}{2\pi} 2\pi\rho \max_{|w-b|=\rho} \left| \frac{f(w)}{(w-b)^{n+1}} \right| = \frac{1}{2\pi} 2\pi\rho \frac{1}{\rho^{n+1}} \max_{|w-b|=\rho} |f(w)| \\ &\le \rho^{-n} M \end{aligned}$$

for any $n \in \mathbb{Z}$ and $0 < \rho < r$. Letting $\rho \downarrow 0$, then $\rho^{-n} \to 0$ if $-n > 0$, so that $c_{-n} = 0$ for $n = 1, 2, \cdots$. Therefore

$$f(z) = \sum_{n=0}^{\infty} c_n (z-b)^n \text{ for } 0 < |z-b| < r.$$

The right-hand side defines a holomorphic function on $D(b,r)$, including b too, so b is removable.

(c, iv) [New, 5 marks] For the last part, we introduce $h(z) = \sin z + \cos z$ whose zeros are isolated. In fact

$$h(z) = \sqrt{2} \sin\left(z + \frac{\pi}{4}\right)$$

so its zeros are $a_n = n\pi - \frac{\pi}{4}$ where $n \in \mathbb{Z}$. Now consider $g(z) = f(z)/h(z)$ for $z \neq a_n$, so that a_n are isolated singularity of g. Now $|g(z)| \leq 1$ for all $z \neq a_n$, thus a_n are all removable, and therefore g is (extended to be) holomorphic in $\mathbb{C}$, and still we have $|g(z)| \leq 1$, thus g is bounded and holomorphic in $\mathbb{C}$, so that g must be constant. It follows that

$$f(z) = Ah(z) = A(\sin z + \cos z)$$

for a constant A, with $|A| \leq 1$.

Q4. (a) [Similar + Book, 10 marks=5+5]

Solution. (a, i) [5 marks] We show that A is open. Suppose $a \in R$, then there is $r > 0$ such that $D(a,r) \subseteq R$, and according to Taylor's expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \text{ for } |z-a| < r$$

where $a_n = f^{(n)}(a)$. If $a \in A$, then all $a_n = 0$ so that $f(z) = 0$ for $z \in D(a,r)$, and therefore $f^{(n)}(z) = 0$ too for all $n \geq 0$ and $z \in D(a,r)$. Therefore A is open. On the other hand for each n, $\left\{z \in R : f^{(n)}(z) = 0\right\}$ is the pre-image of $\{0\}$ under $f^{(n)}$ which is continuous, and therefore is closed. It follows that

$$A = \bigcap_{n=0}^{\infty} \left\{z \in R : f^{(n)}(z) = 0\right\}$$

is closed in R. Therefore A is a closed and open subset of R. Since R is connected, so that A must be empty or $A = R$.

(a, ii) [5 marks] *Identity Theorem.* Suppose f is holomorphic on a connected open set $R \subseteq \mathbb{C}$. Suppose there is a sequence of $z_n \in R$, $z_n \to a$, where $z_n \neq a$, $a \in R$ and $f(z_n) = 0$ for $n = 1, 2, \cdots$, then $f(z) = 0$ for all $z \in R$. [Or other equivalent formulation].

Proof. Let A as defined above. We show that $a \in R$. By Taylor's expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \text{ for } |z-a| < r$$

for some $r > 0$. We claim that all $a_n = 0$. Since $f(a) = \lim_{n\to\infty} f(z_n) = 0$, so $a_0 = 0$. Suppose there is $k \in \mathbb{N}$, such that $a_k \neq 0$, and $a_n = 0$ for $n < k$, then

$$\frac{f(z)}{(z-a)^k} = a_k + a_{k+1}(z-a)^{k+1} + \cdots$$

for any $0 < |z-a| < r$. In particular

$$0 = \frac{f(z_n)}{(z_n - a)^k} = a_k + a_{k+1}(z_n - a)^{k+1} + \cdots$$

so by letting $n \to \infty$ we obtain that $a_k = 0$ which contradicts to the assumption. Therefore $f^{(k)}(a) = a_k = 0$ for all k. Hence $a \in A$, thus A is non-empty, so that $A = R$, which means that $f(z) = 0$ for every $z \in R$.

(b) [Similar, 5 marks = 2+1+2]

Solution. (b, i) [2 marks] Since $\frac{1}{n} \to 0 \in D$, by Constancy Theorem, $f(z) = 2$ as $f(\frac{1}{2n}) = 2$, and also $f(z) = 0$ as $f\left(\frac{1}{2n+1}\right) = 0$, for every $z \in D$. Therefore there is no such holomorphic function.

(b, ii) [1 mark] $f(z) = \frac{1}{1+z}$ will do, which is unique.

(b, iii) [2 marks] $f(z) = z^2$ will do, also

$$f(z) = z^2 + \sin\frac{\pi}{1-z}$$

works as well. Constancy Theorem does not apply in this case as $\frac{n-1}{n} \to 1$, but 1 does not belong to the open unit disk $D(0,1)$.

(c) [New, 10 marks =4+3+3]

Solution. (c, i) [4 marks] If a is an isolated singularity for f, then there is $r > 0$, we have Laurent's expansion about a in the following form

$$f(z) = \sum_{n=1}^{\infty} c_{-n}(z-a)^{-n} + \sum_{n=1}^{\infty} c_n(z-a)^n \text{ for } 0 < |z-a| < r$$

here the coefficient

$$c_n = \frac{1}{2\pi i}\int_{C(a,\rho)} \frac{f(w)}{(w-a)^{n+1}} dw$$

where $0 < \rho < r$. The coefficient of $(z-a)^{-1}$, that is,

$$c_{-1} = \frac{1}{2\pi i}\int_{C(a,\rho)} f(z)dz \qquad (1)$$

is called the residue of f at a. [It is an acceptable answer that the residue of f at a is given by (1).]

a is essential if there are infinite many c_{-n} (where $n \in \mathbb{N}$) do not vanish.

(c, ii) [3 marks] By contradiction, if f is bounded, then a is removable and thus not essential. In fact, suppose f is bounded near a, so that there is $0 < \delta_0 < \delta$ and $M > 0$ such that $|f(z)| \leq M$ for $0 < |z-a| < r$. Then, by part a), for $n \geq 1$, $0 < \rho < r$, we have by Estimation Lemma

$$|c_{-n}| \leq \frac{1}{2\pi} 2\pi\rho \max_{|w-a|=\rho} \left| \frac{f(w)}{(w-a)^{-n+1}} \right| \leq \rho^n M$$

by letting $\rho \downarrow 0$ we may conclude that all c_{-n} vanish for $n = 1, 2, \cdots$, so a is not essential by definition.

(c, iii) [3 marks] For every complex number C, a is an essential singularity of $f(z) - C$ too. If there is a sequence $z_n \to a$ such that $f(z_n) - C = 0$, then we are done. Otherwise, there is $r > 0$ such that $f(z) - C \neq 0$ has on $D(a, r) \setminus \{a\}$ for some $r > 0$. Now consider the function $h(z) = \frac{1}{f(z)-C}$ for $0 < |z-a| < r$, so that a is an isolated singularity. We show that h is unbounded. Suppose h is bounded, then a is a removable singularity of h so that and $\lim_{z\to a} h(z)$ exists. If $h(z) \to 0$, then a is a zero of h. Since h is not identically equal to zero, so that a is an isolated zero. Hence $h(z) = (z-a)^k g(z)$ where $g(z)$ is homolorphic, and has no zero in $|z-a| < r$ for some $r > 0$. Hence

$$f(z) = C + \frac{1}{h(z)} = C + \frac{1}{(z-a)^k} \frac{1}{g(z)}$$

and therefore a is a pole of order k, not an essential one, which is a contradiction. Hence $h(a) \neq 0$. In this case $h \neq 0$ is holomorphic in $D(a, r)$ for some $r > 0$, hence $f(z) = C + \frac{1}{h(z)}$ is holomorphic up to a too, thus a is removable, a contradiction too.

We therefore can conclude that a is an essential singularity of h too. In particular h is unbounded, hence there is z such that $0 < |z-a| < \delta$ but $|h(z)| > \frac{1}{\varepsilon}$, that is $|f(z) - C| < \varepsilon$.

It follows that $\lim_{z\to a} f(z)$ can not exist or equal to ∞.

Q5. (a) [Book+Similar, 9 marks = 3+6]

Solution. The Residue Theorem: Let D be an open subset enclosed by a piece-wise smooth curve C, so $\overline{D} = D \cup C$. Suppose f is holomorphic in D except for finite many isolated singularities $a_1, \cdots, a_m \in D$ and f is at any $z \in C$. Suppose C is orientated positively with respect to D. Then

$$\int_C f(z)dz = 2\pi i \sum_{i=1}^{m} \text{Res}(f, a_i).$$

[3 marks]

Let us do the substitution $z = e^{i\theta}$ with $\theta : 0 \to 2\pi$ in the integral on the left-hand side, which is denoted by I for simplicity. Then $d\theta = dz/(iz)$, $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$ so that I can be turned into the following contour integral

$$\begin{aligned}
I &= \int_{C(0,1)} \frac{1}{1 - p\left(z + \frac{1}{z}\right) + p^2} \frac{dz}{iz} \\
&= \frac{1}{i} \int_{C(0,1)} \frac{1}{z - p(z^2+1) + p^2 z} dz \\
&= \frac{1}{i} \int_{C(0,1)} \frac{1}{(z-p)(1-pz)} dz \\
&= 2\pi \frac{1}{1-pp} = \frac{2\pi}{1-p^2}
\end{aligned}$$

where the last equality follows from the Cauchy formula applying to function $\frac{1}{1-pz}$ at point p, which is holomorphic in $D(0, \frac{1}{|p|})$. [6 marks]

(b) [Similar+New, 8 marks=4+4]

Solution. Let $f(z) = \frac{\ln^2 z}{1+z^2}$ where $\ln z$ is a holomorphic branch to be chosen later. Set up contour as the following. Let $0 < \varepsilon < R$ where $\varepsilon < 1$ and $R > 1$. The contour Γ consists of the upper semi-circle C_R with center zero and radius R, starting at R to $-R$, which has a parameterization $z = Re^{i\theta}$, $\theta : 0 \to \pi$ and $dz = iRe^{i\theta}d\theta$. The second part Γ_2 is the section along the real line $[-R, -\varepsilon]$, and the 3rd part Γ_3 is $[\varepsilon, R]$, both has parameterization $z = x$ and $dz = dx$. The last part is the small semi-circle C_ε^- center at 0 with radius ε, with clock-wise orientation, so that it has parameterization $z = \varepsilon e^{it}$ where $t : \pi \to 0$, and $dz = i\varepsilon e^{it}dt$.

Choose a holomorphic branch $\ln z$ so that it is holomorphic inside the contour, so we may choose

$$\ln z = \ln|z| + i\arg z, \text{ with } -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}.$$

Then f has one simple pole i inside the contour Γ, whose residue

$$\mathrm{Res}(f,i) = \left.\frac{\ln^2 z}{(1+z^2)'}\right|_{z=i} = \frac{\ln^2 i}{2i} = \frac{1}{2i}\left(\frac{\pi}{2}i\right)^2 = -\frac{1}{2\pi i}\frac{\pi^3}{4}$$

so that by Residue Theorem

$$\int_\Gamma f(z)dz = 2\pi i \mathrm{Res}(f,i) = -\frac{\pi^3}{4}.$$

On the other hand,

$$\begin{aligned}
\int_\Gamma f(z)dz &= \int_{\Gamma_2} f(z)dz + \int_{\Gamma_3} f(z)dz + \int_{C_R} f(z)dz + \int_{C_\varepsilon^-} f(z)dz \\
&= \int_{-R}^{-\varepsilon} \frac{\ln^2 x}{1+x^2}dx + \int_\varepsilon^R \frac{\ln^2 x}{1+x^2}dx + \int_{C_R} f(z)dz + \int_{C_\varepsilon^-} f(z)dz \\
&= \int_{-R}^{-\varepsilon} \frac{(\ln|x| + i\pi)^2}{1+x^2}dx + \int_\varepsilon^R \frac{\ln^2 x}{1+x^2}dx + \int_{C_R} f(z)dz + \int_{C_\varepsilon^-} f(z)dz \\
&= 2\pi i \int_\varepsilon^R \frac{\ln x}{1+x^2}dx - \pi^2 \int_\varepsilon^R \frac{1}{1+x^2}dx + 2\int_\varepsilon^R \frac{\ln^2 x}{1+x^2}dx + \int_{C_R} f(z)dz + \int_{C_\varepsilon^-} f(z)dz. \quad (2)
\end{aligned}$$

[4 marks]

While the path integral

$$\int_{C_R} f(z)dz = \int_0^\pi \frac{(\ln R + i\theta)^2}{1+R^2e^{2\theta i}} iRe^{i\theta}d\theta$$

so that

$$\left|\int_{C_R} f(z)dz\right| \le \pi\frac{(\ln R + \pi)^2}{R^2-1}R \to 0 \text{ as } R \uparrow \infty.$$

Similarly

$$\int_{C_\varepsilon^-} f(z)dz = -\int_0^\pi \frac{(\ln\varepsilon + it)^2}{1+\varepsilon^2 e^{2ti}} i\varepsilon e^{it}dt$$

so that

$$\left|\int_{C_\varepsilon^-} f(z)dz\right| \le \pi\frac{(\ln\varepsilon + \pi)^2}{1-\varepsilon^2}\varepsilon = \pi\frac{(\sqrt{\varepsilon}\ln\varepsilon + \pi\sqrt{\varepsilon})^2}{1-\varepsilon^2} \to 0$$

as $\varepsilon \downarrow 0$.

Therefore by letting $\varepsilon \downarrow 0$ then by letting $R \uparrow \infty$ in (2) to obtain

$$2\pi i \int_0^\infty \frac{\ln x}{1+x^2}dx - \pi^2 \int_0^\infty \frac{1}{1+x^2}dx + 2\int_0^\infty \frac{\ln^2 x}{1+x^2}dx = \lim_{R\uparrow\infty}\lim_{\varepsilon\downarrow 0}\int_\Gamma f(z)dz = -\frac{\pi^3}{4}.$$

Therefore we must have, by comparing the real parts of two sides in the above equation,

$$-\pi^2 \int_0^\infty \frac{1}{1+x^2}dx + 2\int_0^\infty \frac{\ln^2 x}{1+x^2}dx = -\frac{\pi^3}{4}$$

so that

$$2\int_0^\infty \frac{\ln^2 x}{1+x^2}dx = \pi^2 \int_0^\infty \frac{1}{1+x^2}dx - \frac{\pi^3}{4} = \pi^2\frac{\pi}{2} - \frac{\pi^3}{4} = \frac{\pi^3}{4}$$

and therefore

$$\int_0^\infty \frac{\ln^2 x}{1+x^2}dx = \frac{\pi^3}{8}.$$

[4 marks]

(c) [Book-New, 8 marks=5+3]

Solution. (c,i) [5 marks] By Taylor's expansion

$$f(z) = \sum_{n=0}^\infty a_n(z-a)^n$$

for all z such that $|z-a| < r$, where $r > 0$ is any such that $D(a,r) \subseteq D(0,1)$. If f is not identically equal to zero, then not all $a_n = 0$, thus, there is m such that $a_m \neq 0$ but $a_l = 0$ for any $l < m$. Hence

$$f(z) = (z-a)^m \sum_{n=k}^\infty a_k(z-a)^{n-m} = (z-a)^m g(z)$$

where $g(z) = \sum_{n=k}^\infty a_n(z-a)^{n-m}$ which is holomorphic and $g(a) \neq 0$. If f is not identically equal to zero, then a is an isolated zero, so that a is an isolated singularity of f'/f. In fact, since $f'(z) = m(z-a)^{m-1}g(z) + (z-a)^m g'(z)$, so that

$$\begin{aligned}\frac{f'(z)}{f(z)} &= \frac{m(z-a)^{m-1}g(z) + (z-a)^m g'(z)}{(z-a)^m g(z)} \\ &= \frac{m}{z-a} + \frac{g'(z)}{g(z)}.\end{aligned}$$

Since $g(a) \neq 0$, so g'/g is holomorphic near a, thus a is a simple pole of f'/f.

(c,ii) [3 marks] Similar to (i), if a is a zero of f, then

$$\begin{aligned}\phi(z)\frac{f'(z)}{f(z)} &= \phi(z)\frac{m(z-a)^{m-1}g(z) + (z-a)^m g'(z)}{(z-a)^m g(z)} \\ &= \frac{m}{z-a}\phi(z) + \phi(z)\frac{g'(z)}{g(z)}.\end{aligned}$$

Since $g(a) \neq 0$ and g and ϕ are holomorphic, so that a is a simple pole of $\phi(z)\frac{f'(z)}{f(z)}$, whose residue equals

$$\begin{aligned}\lim_{z\to a}(z-a)\phi(z)\frac{f'(z)}{f(z)} &= \lim_{z\to a} m\phi(z) + \lim_{z\to a}(z-a)\phi(z)\frac{g'(z)}{g(z)} \\ &= m\phi(a).\end{aligned}$$

Therefore, according to the Residue Theorem

$$\frac{1}{2\pi i}\int_{C(0,1)} \phi(z)\frac{f'(z)}{f(z)}dz = \sum_{i=1}^{k} \mathrm{Res}(\phi(z)\frac{f'(z)}{f(z)}, a_i) = \sum_{i=1}^{k} m_i\phi(a_i)$$

which completes the proof.

Q6. (a) [Similar, 7 marks=4+3]

Solution. By the basic properties of Möbius transformations, if h maps H onto D, and sends a to 0, then h sends $\bar{a}$ to ∞, so that

$$h(z) = e^{i\theta}\frac{z-a}{z-\bar{a}}$$

where θ is real. The inverse of h maps D onto H. To work out the inverse of h explicitly. [4 marks]

Let $w = h(z)$. Then

$$e^{-i\theta}w = \frac{z-a}{z-\bar{a}}$$

solves z to obtain

$$z = \frac{\bar{a}w - ae^{i\theta}}{w - e^{i\theta}}.$$

[3 marks]

(b) [New, 6 marks] Let $w = \varphi(z)$, where $|\alpha| \neq 1$. Then

$$\begin{aligned}|w|^2 - 1 &= \frac{z-\alpha}{1-\overline{\alpha}z}\frac{\bar{z}-\bar{\alpha}}{1-\alpha\bar{z}} - 1 \\ &= \frac{(1-|\alpha|^2)(|z|^2-1)}{1-\bar{\alpha}z-\alpha\bar{z}+|\alpha|^2|z|^2} \\ &= \frac{1-|\alpha|^2}{|1-\overline{\alpha}z|^2}(|z|^2-1)\end{aligned}$$

so that, if $|\alpha| < 1$, then $|w| < 1$ if and only if $|z| < 1$. [3 marks]

If $|\alpha| > 1$, then $|w| > 1$ if and only if $|z| < 1$. That is, if $|\alpha| < 1$, then $\varphi(D) = D$, and if $|\alpha| > 1$ then $\varphi(D) = D^c$. [3 marks]

$(\mathbb{C}\setminus D)$

(c) [Similar, 12 marks]

Solution. The circle $|z| = 1$ and the real line $\text{Im}z = 0$ intersect at -1 and 1, thus we first apply the Möbius transformation f_1 which sends 1 to 0, -1 to ∞, that is

$$f_1(z) = \frac{z-1}{z+1}.$$

Now $i \in$upper semi-circle $|z| = 1$ with $\text{Im}z > 0$, and

$$f_1(i) = \frac{i-1}{i+1} = \frac{(i-1)(-i+1)}{2} = i$$

so the unit circle is mapped to the line $\text{Re}z = 0$, i.e. the y-axis.

$0 \in \text{Im}z = 0$, and $f(0) = -1$, so f_1 maps the real line to real line.

Choose a point say $i/2$ inside the domain R. Since

$$f(\frac{i}{2}) = \frac{i-2}{i+2} = \frac{(i-2)(-i+2)}{5} = \frac{-3+4i}{5}$$

we thus may conclude that

$$f_1(R) = \{z = x+iy : y > 0, x < 0\}.$$

[6 marks]

Let $f_2(z) = e^{-\frac{\pi}{2}i}z = -iz$, so that

$$f_2 \circ f_1(R) = \{z = x+iy : y > 0, x > 0\}$$

and then apply $f_3(z) = z^2$, so that

$$f_3 \circ f_2 \circ f_1(z) = \left(-i\frac{z-1}{z+1}\right)^2 = -\left(\frac{z-1}{z+1}\right)^2$$

maps R to H. Therefore, by part 1)

$$f(z) = e^{i\theta}\frac{-\left(\frac{z-1}{z+1}\right)^2 - a}{-\left(\frac{z-1}{z+1}\right)^2 - \bar{a}}$$

maps R one to one onto the unit disk D, where θ is any real, $a \in H$ can be any. [6 marks]