A4 Integration

Solution of Question 1. (a) [*Book work*] $(\Omega, \mathcal{F})$ is called a measurable space if $\mathcal{F}$ is a σ-algebra of some subsets of Ω, here $\mathcal{F}$ is a σ-algebra on Ω if Ω and empty set $\emptyset$ belong to $\mathcal{F}$, if $A \in \mathcal{F}$ then so does A^c, and if $A_n \in \mathcal{F}$ (where $n = 1, 2, \cdots$) then the countable union $\cup_{n=1}^{\infty} A_n \in \mathcal{F}$ [**1 mark**]. A mapping $\mu : \mathcal{F} \to [0, \infty]$ is a measure on a measurable space $(\Omega, \mathcal{F})$ if $\mu(\emptyset) = 0$, $\mu(A) \leq \mu(B)$ whenever $A, B \in \mathcal{F}$ and $A \subset B$, and μ is countably additive: if $A_n \in \mathcal{F}$ are disjoint for $n = 1, 2, \cdots$, then $\mu\left(\cup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$ [**1 mark**].

Suppose $E \in \mathcal{F}$, then $E \in \mathcal{F}_E$ and $\emptyset \in \mathcal{F}_E$. If $A \in \mathcal{F}_E$, then $A \in \mathcal{F}$ and $A \subset E$, then $E \setminus A = E \cap A^c \in \mathcal{F}$ as $A^c = \Omega \setminus A \in \mathcal{F}$, thus $E \setminus A \in \mathcal{F}_E$. Suppose now $A_n \subset E$ and $A_n \in \mathcal{F}$, then $\cup_{n=1}^{\infty} A_n \in \mathcal{F}$ and $\cup_{n=1}^{\infty} A_n \subset E$, hence $\cup_{n=1}^{\infty} A_n \in \mathcal{F}_E$. By definition, $\mathcal{F}_E$ is a σ-algebra on E [**2 marks**].

We now prove that ν is a measure on $(E, \mathcal{F}_E)$. Clearly $v(\emptyset) = \mu(\emptyset) = 0$, and if $A \subset B \subset E$ and $A, B \in \mathcal{F}$ then $\nu(A) = \mu(A) \leq \mu(B) = \nu(B)$. Suppose $\{A_n\}$ is a disjoint sequence of elements in $\mathcal{F}_E$, then since $\mathcal{F}_E \subset \mathcal{F}$, and $\cup_{n=1}^{\infty} A_n \in \mathcal{F}_E$ we have by the countable additivity of μ

$$\nu(\cup_{n=1}^{\infty} A_n) = \mu(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n) = \sum_{n=1}^{\infty} \nu(A_n)$$

which shows that ν is a measure on $(E, \mathcal{F}_E)$ [**2 marks**].

(b) [*Book work*] Let $E_1 = A_1$ and $E_n = A_n \setminus A_{n-1}$ for $n \geq 2$, $E_n \in \mathcal{F}$ are disjoint (for $n = 1, 2, \cdots$). More over $\cup_{k=1}^{n} E_k = \cup_{k=1}^{n} A_k = A_n$ for $n = 1, 2, \cdots$, and $\cup_{k=1}^{\infty} E_k = \cup_{k=1}^{\infty} A_k$, thus, by countable additivity of μ

$$\begin{aligned} \mu\left(\cup_{k=1}^{\infty} A_k\right) &= \mu\left(\cup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu\left(E_k\right) = \lim_{n \to \infty} \sum_{k=1}^{n} \mu\left(E_k\right) \\ &= \lim_{n \to \infty} \mu\left(\cup_{k=1}^{n} E_k\right) = \lim_{n \to \infty} \mu\left(A_n\right) \end{aligned}$$

[**3 marks**]. Apply what we proved to the increasing sequence $B_1 \setminus B_n$ and use the fact that

$$\cup_{k=1}^{\infty} (B_1 \setminus B_k) = B_1 \setminus \cap_{k=1}^{\infty} B_k$$

we obtain

$$\mu\left(B_1 \setminus \cap_{k=1}^{\infty} B_k\right) = \lim_{n \to \infty} \mu\left(B_1 \setminus B_n\right).$$

Since $\mu(B_1) < \infty$ so that

$$\mu\left(B_1 \setminus \cap_{k=1}^{\infty} B_k\right) = \mu\left(B_1\right) - \mu(\cap_{k=1}^{\infty} B_k)$$

and

$$\mu\left(B_1 \setminus B_n\right) = \mu\left(B_1\right) - \mu(B_n)$$

as $B_n \subset B_1$ and $\cap_{k=1}^{\infty} B_k \subset B_1$, which yields that $\mu\left(\cap_{k=1}^{\infty} B_k\right) = \lim_{n \to \infty} \mu\left(B_n\right)$ [**4 marks**].

(c) [*New*] Since $h(x) \geq m(\emptyset) \geq 0$, and $h(x) \leq m(E) < \infty$, so h is a real valued, clearly increasing function on $\mathbb{R}$. Suppose $x \leq y$, then

$$E \cap (-\infty, y] = (E \cap (-\infty, x]) \cup (E \cap (x, y])$$

and

$$(E \cap (-\infty, x]) \cap (E \cap (x, y]) = \emptyset$$

so by the additivity of the Lebesgue measure m we have

$$m(E \cap (-\infty, y]) = m\left(E \cap (-\infty, x]\right) + m(E \cap (x, y])$$

that is

$$h(y) = h(x) + m(E \cap (x, y])$$

so that, for $y \geq x$,

$$0 \leq h(y) - h(x) = m(E \cap (x, y]) \leq m((x, y]) = y - x$$

it follows that $|h(y) - h(x)| \leq |y - x|$ for any x, y, thus h is uniformly continuous: for every $\varepsilon > 0$ choose $\delta = \varepsilon$, then $|h(y) - h(x)| < \varepsilon$ as long as $|y - x| < \delta$ [**4 marks**].

Now

$$h(n) = m(E \cap (-\infty, n]) \uparrow m(E) \text{ as } n \uparrow \infty$$

and

$$h(-n) = m(E \cap (-\infty, -n]) \downarrow m(\emptyset) = 0 \text{ as } n \uparrow \infty$$

by 2). Thus, for any $c \in (0, m(E))$ there is n_1 and n_2 such that $h(-n_1) < \frac{c}{2}$ and $h(n_2) > c + \frac{m(E)-c}{2}$ [**4 marks**]. Apply IVT to the continuous function h on $[-n_1, n_2]$, there is $\xi \in [-n_1, n_2]$ such that $h(\xi) = m(E \cap (-\infty, \xi])$, so that $A = E \cap (-\infty, \xi]$ which is Lebesgue measurable will do [**4 marks**].

Solution of Question 2: (a) [*Similar*] f is continuous on $(0, \infty)$ so is measurable, on $(0, 1]$ we have $|f(x)| \leq \frac{x}{x^q} = \frac{1}{x^{q-1}}$. If $q - 1 < 1$ i.e. $q < 2$, $\frac{1}{x^{q-1}}$ is integrable on $(0, 1]$ [**3 marks**]. Now consider f on $[1, \infty)$. In this case, $|f| \geq 1_{[\pi, n\pi]} \frac{|\sin x|}{x^q}$ for every $n = 1, 2, \cdots$, so that

$$\begin{aligned} \int_1^\infty |f(x)| dx &\geq \int_\pi^{n\pi} \frac{|\sin x|}{x^q} dx = \sum_{k=1}^{n-1} \int_{k\pi}^{(k+1)\pi} \frac{|\sin x|}{x^q} dx \\ &= \sum_{k=1}^{n-1} \int_0^\pi \frac{\sin x}{(k\pi + x)^q} dx \geq \frac{\int_0^\pi \sin x dx}{\pi^q} \sum_{k=1}^{n-1} \frac{1}{\pi^q (k+1)^q}. \end{aligned}$$

Since $\sum_{k=1}^\infty \frac{1}{k^q} = \infty$ (divergent), so that, by letting $n \to \infty$ we obtain $\int_1^\infty |f(x)| dx = \infty$, so that $|f|$ is not Lebesgue integrable, hence f is not Lebesgue integrable either [**3 marks**].

(b) [Similar] $(x, y) \to h_y(x)$ is continuous in $(0, \infty) \times [0, \infty)$ so that it is measurable (and h_y is measurable for every $y \geq 0$ fixed). For any fixed $y \geq 0$, $|h_y(x)| \leq \frac{y}{\sqrt{x}}$ on $(0, 1]$, so by comparison to the integrable function $\frac{1}{\sqrt{x}}$ on $(0, 1]$ we may conclude that $x \to h_y(x)$ is integrable on $(0, 1]$. While $|h_y(x)| \leq \frac{y}{x\sqrt{x}}$ on $(1, \infty)$, again by comparison (to the integrable function $\frac{1}{x^{\frac{3}{2}}}$ on $(1, \infty)$), h_y is integrable. Putting together, we deduce that h_y is integrable on $(0, \infty)$ [**1 mark**].

For every $A > 0$

$$|h_y(x)| \leq \frac{A}{\sqrt{x}} 1_{(0,1]}(x) + \frac{A}{x\sqrt{x}} 1_{(1,\infty)}(x) \equiv g(x)$$

and for any $y_0 \in [0, A)$ we have $h_y(x) \to h_{y_0}(x)$ for all x, thus by the theorem of taking limit under integration, we have

$$\lim_{y \to y_0} F(y) = \int_0^\infty h_{y_0}(x) dx = F(y_0)$$

which implies that F is continuous on $[0, A)$ for any $A > 0$. Hence F is continuous on $[0, \infty)$ [**2+4 marks – where 2 marks the theorem quoted**].

We next show F is differentiable on $(0, \infty)$. We have

$$\frac{\partial}{\partial y} h_y(x) = e^{-x} \frac{\cos(xy)}{\sqrt{x}}$$

for all $x > 0$ and $y > 0$, moreover we have the following simple estimate:

$$\left|\frac{\partial}{\partial y}h_y(x)\right| \leq \frac{e^{-x}}{\sqrt{x}}$$

for all $y > 0$ and $x > 0$. $x \to \frac{e^{-x}}{\sqrt{x}}$ is integrable by comparison: for $x > 0$

$$\frac{e^{-x}}{\sqrt{x}} \leq \frac{1}{\sqrt{x}}1_{(0,1]}(x) + e^{-x}1_{(1,\infty)}(x),$$

thus, by the theorem of differentiating functions under integration, $F'(y)$ exists for all $y > 0$ and

$$F'(y) = \int_0^\infty \frac{\partial}{\partial y}h_y(x)dx = \int_0^\infty e^{-x}\frac{\cos(xy)}{\sqrt{x}}dx.$$

[**2+5 marks, where 2 marks for the theorem quoted**]

Theorem 1. Let E be measurable, and $J \subset \mathbb{R}$ be an interval. For $t \in J$, $f_t : E \to [-\infty, \infty]$ is measurable. Suppose for any $t_0 \in J$, $f_t \to f_{t_0}$ almost surely on E, and there is $g \in L^1(E)$ such that $|f_t(x)| \leq g(x)$ almost surely on E for all $t \in J$. Then $f_t \in L^1(E)$ and $F(t) = \int_E f_t$ is continuous on J.

Theorem 2. Let E be a measurable set, and $J \subset \mathbb{R}$ be an interval. For each $t \in J$, $f_t : E \to \mathbb{R}$ is measurable, and the following conditions are satisfied: for every $t \in J$, $f_t \in L^1(E)$, and define $F(t) = \int_E f_t$ for $t \in J$, for every $x \in E$, the partial derivative

$$\frac{\partial}{\partial t}f_t(x) = \lim_{h\to 0}\frac{f_{t+h}(x) - f_t(x)}{h}$$

exists for every $t \in J$ (here the limit runs over $h \to 0$ such that $t + h \in J$), and there is a control function $g \in L^1(E)$ such that

$$\left|\frac{\partial}{\partial t}f_t(x)\right| \leq g(x)$$

almost surely on E for all $t \in J$. Then F is differentiable on J and

$$F'(t) = \int_E \frac{\partial}{\partial t}f_t\ .$$

(c) [*New*] f_n are measurable as f_n are continuous on $(0, \infty)$. We can show for example by L'Hopital rule that $f_n(x) \to 0$ as $n \to \infty$ for every $x > 0$. In order to apply CDT to conclude the limit, we need to find a control function. If $x \in (0, 1]$, then

$$0 \leq f_n(x) \leq \frac{\ln(1+n)}{n}\frac{\sin x}{x^q}.$$

Since $\sin x \leq x$ and $\frac{\ln(1+n)}{n} \to 0$, thus the sequence $\{\frac{\ln(1+n)}{n} : n = 1, 2, \}$ is bounded, hence there is a constant $C_1 > 0$ such that $\frac{\ln(1+n)}{n} \leq C_1$ for all x. Thus

$$0 \leq f_n(x) \leq C_1\frac{1}{x^{q-1}}.$$

Since $q - 1 < 1$ so $C_1\frac{1}{x^{q-1}}$ is integrable on $(0, 1]$. Now we consider $x > 1$. Then

$$\begin{aligned}\frac{\ln(x+n)}{n} &= \frac{\ln n + \ln(\frac{x}{n}+1)}{n} \leq \frac{\ln n + \ln(x+1)}{n} \\ &\leq C_1 + \ln(x+1)\end{aligned}$$

thus

$$|f_n(x)| \le \frac{C_1}{x^q} + \frac{\ln(x+1)}{x^q}$$

for all $x \ge 1$. $\frac{C_1}{x^q}$ is integrable on $(1,\infty)$ as $q > 1$, we show $\frac{\ln(x+1)}{x^q}$ is integrable on $(1,\infty)$ as well. In fact $q > 1$ so we may choose $q > p > 1$ and write

$$\frac{\ln(x+1)}{x^q} = \frac{\ln(x+1)}{x^{q-p}} \frac{1}{x^p}$$

for $x \ge 1$. Let $u(x) = \frac{\ln(x+1)}{x^{q-p}}$. Then u is continuous and non-negative, $u(1) = \ln 2$, $u(x) \to 0$ as $x \to \infty$, thus u must be bounded on $[1,\infty)$. Thus there is $C_2 > 0$ such that $\frac{\ln(x+1)}{x^{q-p}} \le C_2$ for all $x \ge 1$. Thus

$$|f_n(x)| \le \frac{C_1}{x^q} + \frac{C_2}{x^p}$$

for $x \ge 1$. Since both $\frac{C_1}{x^q}$ and $\frac{C_2}{x^p}$ are Lebesgue integrable on $[1,\infty)$. Hence, by DCT we have

$$\lim_{n\to\infty} \int_0^\infty f_n(x)dx = \int_0^\infty \lim_{n\to\infty} f_n(x)dx = 0.$$

DCT: Let E be a measurable set, $f_n : E \to [-\infty,\infty]$ be measurable, and $f = \lim_{n\to\infty} f_n$ almost everywhere on E. Suppose that there is $g \in L^1(E)$ such that $|f_n(x)| \le g(x)$ for almost all $x \in E$. Then f and f_n are integrable, and $\int_E f = \lim_{n\to\infty} \int_E f_n$. [**1+4 marks − 1 marks given to the theorem quoted**]

Solution of Question 3. (a) [*Similar*] f is continuous at $(x,y) \neq (0,0)$ so it is measurable. For $y \neq 0$, $x \to f(x,y)$ is continuous on $[-1,1]$, so it is Riemann integrable, thus must be Lebesgue integrable. The function is odd, so that

$$\int_{-1}^1 f(x,y)dx = 0$$

for all $y \neq 0$. Hence

$$\int_{-1}^1 \left(\int_{-1}^1 f(x,y)dx \right) dy = 0.$$

By symmetry, we also have

$$\int_{-1}^1 \left(\int_{-1}^1 f(x,y)dy \right) dx = 0.$$

[**3 marks**]. The function is not integrable, as it is not integrable on $(0,1)\times(0,1)$ on which f is non-negative. In fact

$$\int_0^1 \frac{xy}{(x^2+y^2)^3}dx = \frac{y}{2}\frac{1}{1-3}\frac{1}{(x^2+y^2)^2}\bigg|_0^1 = \frac{1}{4}\frac{1}{y^3} - \frac{1}{4}\frac{y}{(1+y^2)^2}$$

which is not integrable on $(0,1)$. According to Tonelli's theorem, f can not be integrable on $(0,1)\times(0,1)$, so neither on $[-1,1]\times[-1,1]$. [**2+3 marks − 2 marks given to Tonelli's theorem**].

(b) [*Similar - New*] f is measurable. Let us show f is integrable on $\mathbb{R}^2$ by using Tonelli's theorem. To this end, consider the iterated integral

$$\int_{-\infty}^\infty \left(\int_{-\infty}^\infty |f(x,y)|dx \right) dy = \int_{-\infty}^\infty \left(\int_{-\infty}^\infty \frac{|\sin(x-y)|}{|x-y|^{3/2}} |g(y)|dx \right) dy.$$

Since

$$\int_{-\infty}^{\infty} \frac{|\sin(x-y)|}{|x-y|^{3/2}} |g(y)| dx = |g(y)| \int_{-\infty}^{\infty} \frac{|\sin(x-y)|}{|x-y|^{3/2}} dx.$$

By making change of variable $t = x - y$, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{|\sin(x-y)|}{|x-y|^{3/2}} dx &= \int_{-\infty}^{\infty} \frac{|\sin t|}{|t|^{3/2}} dt = 2\int_{0}^{\infty} \frac{|\sin t|}{t^{3/2}} dt \\
&= 2\int_{0}^{1} \frac{|\sin t|}{t^{3/2}} dt + 2\int_{1}^{\infty} \frac{|\sin t|}{t^{3/2}} dt \\
&\leq 2\int_{0}^{1} \frac{1}{\sqrt{t}} dt + 2\int_{1}^{\infty} \frac{1}{t^{3/2}} dt \\
&= 4 + 4 = 8
\end{aligned}$$

so that

$$\int_{-\infty}^{\infty} |f(x,y)| dx \leq 8|g(y)|$$

for all y. Since g is integrable, so is $|g|$, hence

$$\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |f(x,y)| dx \right) dy \leq 8 \int_{-\infty}^{\infty} |g(y)| dy < \infty.$$

Thus, according to Tonelli's theorem, f is integrable on $\mathbb{R}^2$ [**10 marks**], and Fubini's theorem applies (to both f and $|f|$). It follows that

$$\begin{aligned}
\left| \int_{\mathbb{R}^2} f \right| &\leq \int_{\mathbb{R}^2} |f| = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |f(x,y)| dx \right) dy \\
&\leq 8 \int_{-\infty}^{\infty} |g(y)| dy.
\end{aligned}$$

Fubini's theorem. Let $A, B \in \mathcal{M}_{\mathrm{Leb}}$ [so $A \times B \in \mathcal{M}_{\mathrm{Leb}}(\mathbb{R}^2)$] and $f \in L^2(A \times B)$. Then for almost all $y \in B$, $f_y \in L^1(B)$, where $f_y(x) = f(y,x)$ for $x \in A$, so $F(y) = \int_A f_y$ is well defined for almost all $y \in B$, and F is integrable on B (so in particular, F is Lebesgue measurable), and $\int_B F = \int_{A \times B} f$. Therefore

$$\int_B \left(\int_A f(x,y) \mathrm{d}x \right) \mathrm{d}y = \int_A \left(\int_B f(x,y) \mathrm{d}y \right) \mathrm{d}x = \int_{A \times B} f(x,y) \mathrm{d}x \mathrm{d}y.$$

[**3+4 marks – 3 marks given to Fubini's theorem**].

Tonelli's theorem. Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is measurable, and suppose either of the repeated integrals exists and is *finite*:

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x,y)| \mathrm{d}x \right) \mathrm{d}y, \quad \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x,y)| \mathrm{d}y \right) \mathrm{d}x.$$

Then $f \in L^1(\mathbb{R}^2)$, so that Fubini's theorem is applicable to both f and $|f|$.