A4 Integration Exam 2016 TT

Q1. *Solution of part (a)*
[6 marks Book]

f is Lebesgue measurable on E if for every Borel subset $G \in \mathcal{B}(\mathbb{R})$, $f^{-1}(G) = \{x \in E : f(x) \in G\}$ belongs to $\mathcal{M}_{\text{Leb}}$ (in the case that f takes values in $[-\infty, \infty]$, we require in addition that $f^{-1}(\infty)$ and $f^{-1}(-\infty)$ belongs to $\mathcal{M}_{\text{Leb}}$ as well).

h is Borel measurable (or called Borel function) if for every Borel subset G, $h^{-1}(G)$ is again Borel measurable.

From the lectures, we know that continuous functions on $\mathbb{R}$ are Borel measurable.

Let $g = h \circ f$, and let G be Borel measurable subset. Then $g^{-1}(G) = f^{-1}(h^{-1}(G))$. Since h is Borel measurable, so that $h^{-1}(G)$ is Borel too. Since f is Lebesgue measurable, so that $f^{-1}(h^{-1}(G))$ belongs to $\mathcal{M}_{\text{Leb}}$. Therefore, by definition, $g = h \circ f$ is Lebesgue measurable.

Now $|f|$, f^+ and f^- can be written as $h \circ f$ with $h(x) = |x|$, $\max\{x, 0\}$, and $h(x) = \max\{-x, 0\}$ which are continuous so Borel measurable. Hence $|f|$, f^+ and f^- are Lebesgue measurable.

Solution of part (b)

(i) [*1 mark, Book*] f is Lebesgue integrable if $\int_E |f| dm < \infty$, which equivalent to say that both $\int_E f^+ dm < \infty$ and $\int_E f^- dm < \infty$.

~~[*2 marks, Book*]~~ *MCT*. If $f_n : E \to [0, \infty]$ is a sequence of measurable function, and $f_n \uparrow$ (f_n is increasing in n almost everywhere), then

$$\int_E \lim_{n\to\infty} f_n dm = \lim_{n\to\infty} \int_E f_n dm.$$

Not part of Q1

7 marks

(ii) [~~5 marks~~, *New*] Suppose f is Lebesgue integrable, then f is finite almost everywhere. Consider $g_N = |f| 1_{\{|f| \le N\}}$. Then g_N are measurable, non-negative, and $N \to g_N$ is increasing. Since $|f| < \infty$ almost everywhere, so that $g_N \uparrow |f|$ and therefore

$$\int_{E \cap \{|f| \le N\}} |f| dm = \int_E g_N dm \to \int_E |f| dm.$$

Hence

$$\lim_{N\to\infty} \int_{E\cap\{|f|>N\}} |f| dm = \lim_{N\to\infty} \left(\int_E |f| dm - \int_{E\cap\{|f|\le N\}} |f| dm \right) = 0.$$

(iii) [*5 marks, New*] The L^p-norm $\|f\|_p = \left(\int_E |f|^p dm\right)^{1/p}$ if $\int_E |f|^p < \infty$, otherwise $\|f\|_p = \infty$. Suppose $\sup_n \|f_n\|_p < \infty$ for some $p > 1$. Then $|f_n| \le \frac{1}{N^{p-1}} |f_n|$ on $E \cap \{|f_n| > N\}$ and therefore

$$0 \le \sup_n \int_{E\cap\{|f_n|>N\}} |f_n| dm \le \frac{1}{N^{p-1}} \sup_n \|f_n\|_p^p.$$

Since $p > 1$ so that

$$\frac{1}{N^{p-1}} \sup_n \|f_n\|_p^p \to 0 \text{ as } N \to \infty.$$

Solution of part (c)

[6 marks, Similar] The function $\frac{\cos x}{\sqrt{x}}$ is not integrable on $(0, \infty)$. In fact f is measurable, but

$$\int_\pi^\infty |f(x)| dx = \sum_{n=1} \int_{n\pi}^{(n+1)\pi} \frac{|\cos x|}{\sqrt{x}} dx \geq \sum_{n=1} \int_{n\pi}^{(n+1)\pi} \frac{|\cos x|}{\sqrt{(n+1)\pi}} dx = \sum_{n=1} \int_0^\pi \frac{|\cos x|}{\sqrt{(n+1)\pi}} dx = \infty$$

so $\frac{\cos x}{\sqrt{x}}$ is not integrable on (π, ∞), thus not integrable on $(0, \infty)$ either.

Q2. *Solution of part (a)*

MCT (see Q1 b(i))

[*2 marks, Book*] DCT: If f_n defined on E are measurable, $f_n \to f$ almost everywhere on E, and there is ain integrable function g on E such that $|f_n| \leq g$ almost everywhere on E. Then f_n, f are integrable on E and

$$\int_E f dm = \lim_{n \to \infty} \int_E f_n dm.$$

[*6 marks, Similar*] Both function f and g are continuous on $(0, 1)$ and thus are measurable, and both are non-negative on $(0, 1)$. Moreover

$$\frac{\ln x}{x - 1} = \sum_{n=0}^\infty -x^n \ln x$$

where $-x^n \ln x$ are non-negative and measurable of course on $(0, 1)$, hence by MCT for series we hav

$$\int_0^1 \frac{\ln x}{x - 1} dx = \sum_{n=0}^\infty \int_0^1 -x^n \ln x dx.$$

We show that each $-x^n \ln x$ is integrable in $(0, 1)$. In fact for $N = 1, 2, \cdots$, $-x^n \ln x$ is continuous on $[\frac{1}{N}, 1]$, so that is Riemann integrable on this interval, by performing integration by parts we obtain

$$\int_{1/N}^1 -x^n \ln x dx = \frac{1}{n+1} \left(\frac{1}{N}\right)^{n+1} \ln \frac{1}{N} + \frac{1}{n+1} \int_{1/N}^1 x^n dx$$

so that, by MCT (applying to $-1_{[1/N,1]} x^n \ln x$), we obtain that

$$\int_0^1 -x^n \ln x dx = \lim_{N \to \infty} \frac{1}{n+1} \left(\frac{1}{N}\right)^{n+1} \ln \frac{1}{N} + \frac{1}{n+1} \int_{1/N}^1 x^n dx = \frac{1}{(n+1)^2}.$$

It follows that

$$\int_0^1 \frac{\ln x}{x - 1} dx = \sum_{n=0}^\infty \int_0^1 -x^n \ln x dx = \sum_{n=0}^\infty \frac{1}{(n+1)^2} < \infty$$

so that $\frac{\ln x}{x-1}$ is integrable and

$$\int_0^1 \frac{\ln x}{x-1}dx = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

[*2 marks, Similar*] By making change of variable $x \to 1-x$, we conclude that

$$\int_0^1 \frac{\ln(1-x)}{x}dx = \int_0^1 \frac{\ln x}{1-x}dx = -\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

[2 marks, Book] Change of variable for Lebesgue integrals: if $\varphi : [a,b] \to [c,d]$ is differentiable, strictly increasing or decreasing such that $\varphi(a) = c$, $\varphi(b) = d$. Then a measurable function f is integrable on (c,d) if and only if $f \circ \varphi \varphi'$ is integrable on (a,b), and

$$\int_a^b f(\varphi(x))\varphi'(x)dx = \int_c^d f(x)dx.$$

Solutions of part (b)

[*3 marks, Similar*] f is continuous except at $x = 0$, so it is measurable for every y. Moreover $|\sin(xy)| \le |xy|$, thus

$$|f(x,y)| \le |y|\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} \le |y|\frac{1}{\sqrt{2\pi}}\frac{1}{1+x^2/2} = |y|\frac{2}{\sqrt{2\pi}}\frac{1}{2+x^2}$$

since $\frac{1}{2+x^2}$ is integrable on $(-\infty,\infty)$, so that by comparison, $x \to f(x,y)$ is integrable for every y. (In fact we thus have shown that $e^{-\frac{x^2}{2}}$ is integrable on $(-\infty,\infty)$). Thus F is well defined.

[*2 marks, Similar*] Clearly

$$\frac{\partial}{\partial y}f(x,y) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\cos(xy)$$

and

$$\left|\frac{\partial}{\partial y}f(x,y)\right| \le \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

for every y and x, so that F is differentiable at any y, and we can differentiate under integral to obtain

$$F'(y) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\cos(xy)dx.$$

[*5 marks, New*] We claim that the n-th derivative $F^{(n)}$ exists for every $n = 1, 2, \cdots$, and

$$F^{(n+1)}(y) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\frac{d^n}{dy^n}\cos(xy)dx$$

for $n = 0, 1, 2, \cdots$. That is, we can differentiate again and again under integration to obtain

$$F^{(2n+1)}(y) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}x^{2n}(-1)^n\cos(xy)dx$$

and

$$F^{(2n+2)}(y) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} x^{2n+1}(-1)^{n+1} \sin(xy)dx.$$

We already proved the case that $n = 0$, so let us assume that $n \geq 1$. In fact

$$\begin{aligned} \left| \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \frac{d^n}{dy^n} \cos(xy) \right| &\leq \frac{1}{\sqrt{2\pi}} |x|^n e^{-\frac{x^2}{2}} \\ &= \frac{1}{\sqrt{2\pi}} |x|^n \frac{1}{1 + \frac{x^2}{2} + \cdots + \frac{1}{n!}\frac{x^{2n}}{2^n} + \cdots} \\ &\leq \frac{1}{\sqrt{2\pi}} |x|^n \frac{(2n)!2^{2n}}{(2n)!2^{2n} + x^{4n}}. \end{aligned}$$

The function

$$\frac{1}{\sqrt{2\pi}} |x|^n \frac{(2n)!2^{2n}}{(2n)!2^{2n} + x^{4n}}.$$

is integrable for any $n = 1, 2, \cdots$ by comparing it to $1/|x|^{3n}$ on $(1, \infty)$, and noticing that it is continuous on $[-1, 1]$. Since

$$\frac{d}{dy} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \frac{d^n}{dy^n} \cos(xy) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \frac{d^{n+1}}{dy^{n+1}} \cos(xy)$$

So that we can differentiate under integration for whatever n, to obtain the formula claimed.

[*3 marks, Book*] Differentiation under integration: Let E be a measurable set, and $J \subset \mathbb{R}$ be an interval. For each $t \in J$, $f_t : E \to \mathbb{R}$ is measurable, and the following conditions are satisfied:

1) for every $t \in J$, $f_t \in L^1(E)$, and define $F(t) = \int_E f_t(x)dx$ for $t \in J$,

2) for every $x \in E$, the partial derivative

$$\frac{\partial}{\partial t} f_t(x) = \lim_{h \to 0} \frac{f_{t+h}(x) - f_t(x)}{h}$$

exists for every $t \in J$ (here the limit runs over $h \to 0$ such that $t + h \in J$), and

3) there is a control function $g \in L^1(E)$ such that

$$\left| \frac{\partial}{\partial t} f_t \right| \leq g$$

almost everywhere on E for all $t \in J$. [Here almost everywhere means that there is a null subset $A \subseteq E$, such that $|\frac{\partial}{\partial t} f(t, x)| \leq g(x)$ for $x \in E \setminus A$ and $t \in J$.]

Then F is differentiable on J and

$$F'(t) = \int_E \frac{\partial}{\partial t} f_t(x)dx .$$

Q3. *Solutions of part (a)*

(i) [*2 marks, Book*] State *Fubini's theorem*: Let $X, Y \in \mathcal{M}_{\text{Leb}}(\mathbb{R})$ [so $X \times Y \in \mathcal{M}_{\text{Leb}}(\mathbb{R}^2)$] and $f \in L^1(X \times Y)$. Then

1) for almost all $y \in Y$, $f_y \in L^1(X)$, where $f_y(x) = f(y,x)$ for $x \in X$, so $F(y) = \int_X f_y(x)dx$ is well defined for almost all $y \in Y$, and

2) F defined in 1) is integrable on Y (so in particular, F is Lebesgue measurable), and

$$\int_Y F(y)dy = \int_{X\times Y} f(x,y)dxdy.$$

Therefore

$$\int_Y \left(\int_X f(x,y)\mathrm{d}x\right)\mathrm{d}y = \int_X \left(\int_Y f(x,y)\mathrm{d}y\right)\mathrm{d}x = \int_{X\times Y} f(x,y)\mathrm{d}x\mathrm{d}y.$$

[1 mark, Book] Tonelli's theorem. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is measurable, and suppose either of the repeated integrals exists and is *finite*, i.e.

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x,y)|\mathrm{d}x\right)\mathrm{d}y < \infty, \quad \text{or / and} \quad \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x,y)|\mathrm{d}y\right)\mathrm{d}x < \infty.$$

Then $f \in L^1(\mathbb{R}^2)$, so that Fubini's theorem is applicable to both f and $|f|$.

(ii) [*3 marks, Similar*] f is continuous except for $(0,0)$, so is measurable. $|f(x,y)|$ is symmetric, so we only need to consider repeated integral

$$J \equiv \int_0^1 \left(\int_0^1 \frac{xy}{(x^2+y^2)^\alpha}dx\right)dy = \int_0^1 \left(\frac{y}{2}\int_0^1 \frac{d(x^2+y^2)}{(x^2+y^2)^\alpha}dx\right)dy.$$

f is integrable, according to Tonelli theorem (together with Fubini) if and only if J is finite.

[3 marks, Similr] If $\alpha = 1$, then

$$J = \int_0^1 \left[\frac{y}{2}\ln(1+y^2) - 2y\ln y\right]dy.$$

Since $y\ln y \to 0$ as $y \downarrow 0$, thus $y\ln y$ is bounded and continuous on $(0,1]$, so must be ~~Riemann-integrable, hence~~ is Lebesgue integrable. While $y\ln(1+y^2)$ is continuous on $[0,1]$, so it is Lebesgue integrable. Hence $J < \infty$ if $\alpha = 1$.

[*3 marks, Similar*] If $\alpha \neq 1$, then

$$J = \frac{1}{2}\frac{1}{1-\alpha}\int_0^1 \left[\frac{y}{(1+y^2)^{\alpha-1}} - \frac{1}{y^{2\alpha-3}}\right]dydy.$$

Here $\frac{y}{(1+y^2)^{\alpha-1}}$ is continuous on $[0,1]$ so integrable on $[0,1]$. While $\frac{1}{y^{2\alpha-3}}$ is integrable on $(0,1)$ if and only if $2\alpha - 3 < 1$, that is $\alpha < 2$. Hence J is finite if and only if $\alpha < 2$. Therefore f is integrable on $(-1,1)\times(-1,1)$ if and only if $\alpha < 2$.

Solution of part (b)

[3 marks, New] The function is measurable and non-negative. Let us compute the

$$\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} g(x,y)dy\right)dx = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} \frac{x^2}{\left(1+|y-x|^2\right)\left(e^{x^2}-1\right)}dy\right)dx$$
$$= \int_{-\infty}^{\infty}\left(\frac{x^2}{e^{x^2}-1}\int_{-\infty}^{\infty}\frac{1}{1+|y-x|^2}dy\right)dx.$$

[*3 marks, New*] By change of variable for Lebesgue integration [quote the same as in Q2]

$$\int_{-\infty}^{\infty} \frac{1}{1+|y-x|^2} dy = \int_{-\infty}^{\infty} \frac{1}{1+|y|^2} dy = \pi$$

so that

$$\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} g(x,y) dy \right) dx = \pi \int_{-\infty}^{\infty} \frac{x^2}{e^{x^2}-1} dx = 2\pi \int_{0}^{\infty} \frac{x^2}{e^{x^2}-1} dx$$

[*7 marks, New*] We only need to show

$$\frac{x^2}{e^{x^2}-1}$$

is integrable on $(0, \infty)$. Since

$$\frac{x^2}{e^{x^2}-1} = \frac{x^2}{x^2 + \frac{x^4}{2} + \cdots} \leq \frac{1}{1+\frac{x^2}{2}}$$

so

$$\int_{0}^{\infty} \frac{x^2}{e^{x^2}-1} dx \leq \int_{0}^{\infty} \frac{1}{1+\frac{x^2}{2}} dx \leq \pi.$$

Therefore

$$\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} g(x,y) dy \right) dx \leq 2\pi^2 < \infty.$$

According to Tonelli theorem, g is integrable on $\mathbb{R}^2$.