A4 Integration: Solutions ~~January 16~~ March 6, 2019

Solution to Q.1 (a) [almost all B] A subset $\mathcal{F} \subseteq \mathcal{P}(\Omega)$ is a σ-algebra if:

- $\emptyset \in \mathcal{F}$,
- If $E \in \mathcal{F}$, then $\Omega \setminus E \in \mathcal{F}$,
- If $E_n \in \mathcal{F}$ for $n = 1, 2, \ldots$, then $\bigcup_{n=1}^{\infty} E_n \in \mathcal{F}$. [2]

$\mathcal{F}_{\mathcal{B}}$ is the unique σ-algebra on Ω satisfying

- $\mathcal{B} \subseteq \mathcal{F}_{\mathcal{B}}$,
- If $\mathcal{F}$ is a σ-algebra on Ω and $\mathcal{B} \subseteq \mathcal{F}$, then $\mathcal{F}_{\mathcal{B}} \subseteq \mathcal{F}$. [2]

- $f^{-1}(\emptyset) = \emptyset \in \mathcal{F}$
- If $f^{-1}(E) \in \mathcal{F}$, then $f^{-1}(\Omega \setminus E) = \Omega \setminus f^{-1}(E) \in \mathcal{F}$
- If $f^{-1}(E_n) \in f^{-1}(\mathcal{F})$, then $\bigcup_n f^{-1}(E_n) = f^{-1}(\bigcup_n E_n) \in \mathcal{F}$. [2]

$f(n) = n$ for $n \leqslant 3$

For $\Omega = \mathbb{N}$, $\mathcal{F} = \{\emptyset, \{3\}, \mathbb{N} \setminus \{3\}, \mathbb{N}\}$, $f(3) = f(4) = 3$, $f(n) = n - 1$ for $n \geqslant 4$. Then $f_*(\mathcal{F}) = \{\emptyset, \{3\}, \mathbb{N}\}$, which is not a σ-algebra. [1]

(b) [all B or minor variants] Let $0 \leqslant a < b \leqslant 1$. Then

$$[a, b) = \bigcup_n [a, b - 2^{-n}] \in \mathcal{F}_{\mathcal{B}_1}, \qquad (a, b] = \bigcup_n [a + 2^{-n}, b], \quad (a, b) = \bigcup_n [a + 2^{-n}, b - 2^{-n}] \in \mathcal{F}_{\mathcal{B}_1}.$$

Hence $\mathcal{B}_2 \subseteq \mathcal{F}_{\mathcal{B}_1}$, so $\mathcal{F}_{\mathcal{B}_1} \subseteq \mathcal{F}_{\mathcal{B}_2}$. Since $\mathcal{B}_1 \subseteq \mathcal{B}_2 \subseteq \mathcal{F}_{\mathcal{B}_2}$, $\mathcal{F}_{\mathcal{B}_1} \subseteq \mathcal{F}_{\mathcal{B}_2}$. [3]

Any singleton $\{a\}$ is either $\bigcap_n [a - 2^{-n}, b]$ or $\bigcap_n [a, b + 2^{-n}]$, and hence in $\mathcal{M}_{\text{Bor}}$. Any countable set is a countable union of singletons and hence is in $\mathcal{M}_{\text{Bor}}$. [1]

(c) [(i)S; rest N] (i) If I is an interval in $[0, 1]$, then $\Phi^{-1}(I)$ is an interval, since Φ is monotonic. Thus the σ-algebra $\mathcal{H}$ contains $\mathcal{B}_2$, and is contained in $\mathcal{M}_{\text{Bor}}$. Since $\mathcal{M}_{\text{Bor}}$ is the smallest such σ-algebra, $\mathcal{H} = \mathcal{M}_{\text{Bor}}$, and hence $\Phi_*(\mathcal{M}_{\text{Bor}}) \subseteq \mathcal{M}_{\text{Bor}}$. For $E \in \mathcal{M}_{\text{Bor}}$, $E = \Phi(\Phi^{-1}(E)) \in \Phi^*(\mathcal{M}_{\text{Bor}})$. [3]

(ii) If $y \in \Phi(E) \cap \Phi(X \setminus E)$, then $y = \Phi(x_1) = \Phi(x_2)$ for distinct x_1, x_2. There are at most countably many such points. [1]

If $\Phi(E) \in \mathcal{M}_{\text{Bor}}$, then $\Phi([0, 1] \setminus E) = ([0, 1] \setminus \Phi(E)) \cup (\Phi(E) \cap \Phi(X \setminus E)) \in \mathcal{M}_{\text{Bor}}$, using (b). [2]

(iii) Trivially, $\emptyset \in \mathcal{G}$.

Let $E \in \mathcal{G}$. Then $X \setminus E \in \mathcal{G}$, by (ii).

Let $E_n \in \mathcal{G}$. Then $\Phi(\bigcup E_n) = \bigcup \Phi(E_n) \in \mathcal{M}_{\text{Bor}}$, so $\bigcup E_n \in \mathcal{G}$. Now $\mathcal{M}_{\text{Bor}} \subseteq \mathcal{G}$. By definition, $\mathcal{G} \subseteq \mathcal{M}_{\text{Bor}}$. [2]

(iv) Let $E \in \mathcal{G}$. Then $\Phi(E) \in \mathcal{M}_{\text{Bor}}$, so $\Phi(X \setminus E) \in \mathcal{M}_{\text{Bor}}$ by (ii). Thus $X \setminus E \in \mathcal{G}$. If $E_n \in \mathcal{G}$ then $\Phi(\bigcup U_n) = \bigcup \Phi(E_n) \in \mathcal{M}_{\text{Bor}}$. So $\mathcal{G}$ is a σ-algebra.

If I is an interval, then $\Phi(I)$ is an interval, so $I \in \mathcal{G}$. Thus $\mathcal{G}$ is a σ-algebra containing $\mathcal{B}_2$ and contained in $\mathcal{M}_{\text{Bor}}$, so $\mathcal{G} = \mathcal{M}_{\text{Bor}}$. If $E \in \mathcal{M}_{\text{Bor}} = \mathcal{G}$, then $E = \Phi(\Phi^{-1}(E)) \in \Phi^*(\mathcal{M}_{\text{Bor}})$. [3]

(v) Let $E \subseteq [0, 1]$. Since $\Phi(C) = [0, 1]$, $E = \Phi(C \cap \Phi^{-1}(E))$. Since C is null, $C \cap \Phi^{-1}(E)$ is null, so $C \cap \Phi^{-1}(E) \in \mathcal{M}_{\text{Leb}}$, and $E \in \Phi^*(\mathcal{M}_{\text{Leb}})$. Thus $\Phi^*(\mathcal{M}_{\text{Leb}}) = \mathcal{P}([0, 1])$. [3]

Solution to Q.2

(a) (i) [B/S] No. Take $f(x) = \sin x$, and $g(x) = (x + 2\pi)^{-1}$ for example. [3]

(ii) [S] The given functions are continuous, hence measurable, and they are dominated by e^{-x}. This is a non-negative measurable function, and, by FTC,

$$\int_0^k \mathrm{e}^{-x}\,\mathrm{d}x = 1 - \mathrm{e}^{-k} \to 1.$$

By the Baby MCT, e^{-x} is integrable over $(0, \infty)$. By comparison, the given functions are also integrable. [2]

(iii) [S]

$$I_0 = \int_0^\infty \mathrm{e}^{-x}\,\mathrm{d}x = \lim_{k\to\infty} \int_0^k \mathrm{e}^{-x}\,\mathrm{d}x = 1,$$
$$J_0 = \mathrm{Re} \int_0^\infty \mathrm{e}^{-x}\mathrm{e}^{ix}\,\mathrm{d}x = \mathrm{Re} \lim_{k\to\infty} \left(\frac{1 - \mathrm{e}^{-k}\mathrm{e}^{ik}}{1 - i}\right) = \mathrm{Re}\left(\frac{1+i}{2}\right) = \frac{1}{2}.$$

Let $n \geqslant 1$, and let I_n^k and J_n^k be the corresponding integrals over $[0, k\pi]$. By integration by parts (noting that $\sin^n k\pi = 0$), we obtain that

$$I_n^k = \int_0^{k\pi} \mathrm{e}^{-x} n \sin^{n-1} x \cos x\,\mathrm{d}x = nJ_{n-1}^k,$$
$$J_n^k = \int_0^{k\pi} \mathrm{e}^{-x}\left(n \sin^{n-1} x \cos^2 x - \sin^n x \sin x\right)\,\mathrm{d}x$$
$$= \int_0^{k\pi} \mathrm{e}^{-x}\left(n \sin^{n-1} x - (n+1)\sin^{n+1} x\right)\,\mathrm{d}x = nI_{n-1}^k - (n+1)I_{n+1}^k.$$

Letting $k \to \infty$ gives the desired equalities. [5]

(b) (i) [S] $1 + \sin x$ and its derivative both vanish at $x = 3\pi/2$, and $|1 + \sin x| < |x - 3\pi/2|$ for all x (for example, by Mean Value Theorem). For $x \in (\pi, 2\pi)$, $\mathrm{e}^{-x}(1 + \sin x)^{-1} \geqslant e^{-2\pi}|x - 3\pi/2|^{-1}$, which is not integrable over $(\pi, 2\pi)$, so $\mathrm{e}^{-x}(1 + \sin x)^{-1}$ is not integrable, by comparison. [3]

(iii) [S]

$$\mathrm{e}^{-x}(1 + r\sin x)^{-1} = \mathrm{e}^{-x}\sum_{n=0}^\infty (-r\sin x)^n = \sum_{n=0}^\infty (-1)^n r^n \mathrm{e}^{-x}\sin^n x = \lim_{m\to\infty} g_m(x),$$

where $g_m(x) = \sum_{n=0}^m (-1)^n r^n \mathrm{e}^{-x}\sin^n x$. Then $|g_m(x)| \leqslant \sum_{n=0}^m r^n \mathrm{e}^{-x} \leqslant (1-r)^{-1}\mathrm{e}^{-x}$, which is integrable over $(0, \infty)$. By the DCT, $\mathrm{e}^{-x}(1 + r\sin x)^{-1}$ is integrable and its integral is

$$\lim_{m\to\infty} \int_0^\infty g_m(x)\,dx = \lim_{m\to\infty} \sum_{n=0}^m (-1)^n r^n I_n = \sum_{n=0}^\infty (-1)^n r^n I_n. \qquad [6]$$

(iv) [N] Suppose that $\sum_{n=0}^\infty (-1)^n I_n$ converges (to a real number). Then, using Abel's continuity theorem, (iii), MCT, and (i),

$$\sum_{n=0}^\infty (-1)^n I_n = \lim_{r\nearrow 1} \sum_{n=0}^\infty (-1)^n r^n I_n = \lim_{r\nearrow 1} \int_0^\infty \mathrm{e}^{-x}(1 + r\sin x)^{-1}\,\mathrm{d}x$$
$$\geqslant \lim_{r\nearrow 1} \int_\pi^{2\pi} \mathrm{e}^{-x}(1 + r\sin x)^{-1}\,\mathrm{d}x = \infty.$$

This is a contradiction. [6]

Solution to Q.3

(a) [all B] Fubini: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be integrable. Then, for almost all y, the function $x \mapsto f(x,y)$ is integrable over $\mathbb{R}$. Moreover, if $F(y)$ is defined (for almost all y) by $F(y) = \int_{\mathbb{R}} f(x,y)\,dx$, then F is integrable, and [3]

$$\int_{\mathbb{R}^2} f(x,y)\,d(x,y) = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x,y)\,dx \right) dy.$$

Tonelli: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a measurable function, and suppose that either of the following repeated integrals is finite:

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x,y)|\,dx \right) dy, \qquad \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x,y)|\,dy \right) dx.$$

Then f is integrable. Hence, Fubini's theorem is applicable to both f and $|f|$. [2]

(b) [all S] Consider the region $u := \{(x,y) : 0 < y < x < 1\}$, where $f(x,y) > 0$. For fixed x consider

$$\int_0^x f(x,y)\,dy = \lim_{n\to\infty} \left(x \log x - \frac{x}{2} \log(x^2 - (x - n^{-1})^2) \right) = \infty.$$

Thus the function $f(x,\cdot)$ is not integrable for any $x \in (0,1)$. By the contrapositive of Fubini's theorem, f is not integrable over U and hence not integrable over $(0,1) \times (0,1)$. [5]

(c) [S] Let $f(x,y) = \mathrm{e}^{-x^2/y}\mathrm{e}^{-y}y^{-1}$ on $\mathbb{R}\times(0,\infty)$. Then f is measurable and non-negative. Moreover, for fixed $y > 0$, putting $x = u\sqrt{y}$,

$$\int_{\mathbb{R}} f(x,y)\,\mathrm{d}x = \int_{\mathbb{R}} \mathrm{e}^{-u^2}\mathrm{e}^{-y}y^{-1/2}\,\mathrm{d}u = \sqrt{\pi}e^{-y}y^{-1/2}.$$

This function of y is integrable over $(0,\infty)$, by comparison with $y^{-1/2}$ on $(0,1)$, and e^{-y} on $(1,\infty)$. By Tonelli, f is integrable; by Fubini, g is integrable over $\mathbb{R}$. [4]

[S/N] Now the function $(x,y) \mapsto \mathrm{e}^{-isx}f(x,y)$ is measurable, and its absolute value is f, so the function is integrable over $(0,\infty) \times \mathbb{R}$. Using Fubini and the given information,

$$\begin{aligned}\widehat{g}(s) &= \int_{\mathbb{R}} \int_0^\infty \mathrm{e}^{-isx} f(x,y)\,\mathrm{d}y\,\mathrm{d}x = \int_0^\infty \int_{\mathbb{R}} \mathrm{e}^{-isx} f(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_0^\infty \mathrm{e}^{-s^2y/4}\mathrm{e}^{-y}y^{-1/2}\,\mathrm{d}y = \int_0^\infty \mathrm{e}^{-(1+s^2/4)y}\sqrt{\pi}\,y^{-1/2}\,\mathrm{d}y.\end{aligned}$$

Making the further change of variable, $u = (1 + s^2/4)y$ gives

$$\widehat{g}(s) = \left(2\sqrt{\pi} \int_0^\infty \mathrm{e}^{-u}u^{-1/2}\,du \right) (4 + s^2)^{-1/2}$$ [5]

(d) [N] Assume that $\widetilde{h}$ is integrable, so $\left|\widetilde{h}\right|$ is integrable. Let $a \in (0,1)$. For fixed $x \in (0,a)$, putting $t = xy$,

$$\int_1^\infty \left|\widetilde{h}(x,y)\right|\,\mathrm{d}y = \left(\int_x^\infty |h(t) - h(2t)|\,dt \right) x^{-1} \geqslant \left(\int_a^\infty |h(t) - h(2t)|\,dt \right) x^{-1}.$$

By Fubini, the left-hand side is integrable over $(0,1)$ and hence over $(0,a)$. By comparison, the left-hand side is integrable over $(0,a)$. But x^{-1} is not integrable over $(0,a)$, so the term $\int_a^\infty |h(t) - h(2t)|dt$ must be 0. This implies that the integrand is 0 a.e. in (a,∞). Since a is arbitrary this implies that $h(t) = h(2t)$ a.e. in $(0,\infty)$.

On the other hand, if $h(t) = h(2t)$ a.e., then $\widetilde{h} = 0$ a.e., and $\widetilde{h}$ is integrable. [6]