Solutions Topology A.5

Problem 1

(a) i) A topological space X is disconnected if there are disjoint open non-empty subsets U and V such that $U \cup V = X$. If X is not disconnected, it is called connected.

A (non-empty) subset A of a topological space X is connected if A with the subspace topology is connected. [B, 2]

ii) Assume that $A \cup B$ is disconnected. Then there are open subsets of X, U, V, such that

$$(A \cup B) \cap U \neq \emptyset,\ (A \cup B) \cap V \neq \emptyset,\ (A \cup B) \cap U \cap V = \emptyset$$

Let $x \in A \cap B$. Then $x \in U$ or $x \in V$. Say $x \in U$. Since A is connected $A \subset U$. For the same reason $B \subset U$. So $(A \cup B) \cap V = \emptyset$, a contradiction. [B, 4]

iii) Assume that $f(A)$ is not connected. Let U and V be open sets such that

$$f(A) \cap V \neq \emptyset,\ f(A) \cap U \neq \emptyset,\ f(A) \cap V \cap U = \emptyset.$$

Then $f^{-1}1(U)$ and $f^{-1}(V)$ are open in X. So, $f^{-1}(U) \cap A$ and $f^{-1}(V) \cap A$ are disjoint and open in A. Since A is connected, one of $f^{-1}(U) \cap A$ and $f^{-1}(V) \cap A$ is empty. Hence, one of $f(A) \cap U$ and $f(A) \cap V$ is empty. So, $f(A)$ is connected. [B, 4]

(b) i) Let $\mathcal{B} = \{U \times V : U \subset \mathcal{T}_X, V \subset \mathcal{T}_Y\}$. Clearly $X \times Y \subset \mathcal{B}$. Also

$$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2 \times V_1 \cap V_2)$$

hence $(U_1 \times V_1) \cap (U_2 \times V_2)$ lies in $\mathcal{B}$. So $\mathcal{B}$ is a basis for a topology. [B, 4]

ii) Assume that X, Y are Hausdorff. Let $(x, y) \neq (x'y')$. Then either $x \neq x'$ or $y \neq y'$. Without loss of generality we assume that $x \neq x'$. Then there exist U, V open disjoint in X with $x \in U, x' \in V$. It follows that $(x, y) \in U \times Y, (x'y') \in V \times Y$ and $U \times Y \cap V \times Y = \emptyset$. So $X \times Y$ is Hausdorff.

Conversely we show that X is Hausdorff: Let $x, x' \in X$ with $x \neq x'$. Given $y \in Y$ there are open sets U_1, U_2 in X, V_1, V_2 in Y such that

$$(x, y) \in U_1 \times V_1,\ (x', y) \in U_2 \times V_2,\ U_1 \times V_1 \cap U_2 \times V_2 = \emptyset.$$

Then $x \in U_1, x' \in U_2$ and $U_1 \cap U_2 = \emptyset$. The same argument applies to Y. [B and S, 4]

iii) We show first a lemma: If C and $A_i, i \in I$ are connected sets in a topological space Z and $A_i \cap B \neq \emptyset$ for all i then $F = C \cup \bigcup_{i \in I} A_i$ is connected. Indeed let U, V be open subsets of Z such that $U \cap V \cap F = \emptyset$ and $F \subset U \cup V$. If $C \cap U \neq \emptyset$ then $C \subset U$ since C is connected. For the same reason $A_i \subset U$ for all i. It follows that $F \subset U$. We argue similarly if $C \cap V \neq \emptyset$. It follows that either $F \cap U$ or $F \cap V$ is empty. So F is connected.

Let $x_0 \in X - A, y_0 \in Y - B$.

$$X \times Y \setminus A \times B = X \times (Y - B) \cup Y \times (X - A)$$

Each set $X \times \{y\}$ with $y \in Y - B$ is connected and intersects $\{x_0\} \times Y$. So by the lemma $X \times (Y - B) \cup \{x_0\} \times Y$ is connected. By the same argument $Y \times (X - A) \cup X \times \{y_0\}$ is connected. (x_0, y_0) lies in both these sets so their union which is equal to $X \times (Y - B) \cup Y \times (X - A)$ is connected by part (a ii). [N, 4]

iv) Let $A \subset X$ be finite and let let $a \in A$. If $A = \{a\}$ clearly A is connected. Otherwise if $B = A \setminus \{a\}$, then $X \setminus B, X \setminus \{a\}$ are open sets, $A \subset (X \setminus B) \cup (X \setminus \{a\})$ and $(X \setminus B) \cap (X \setminus \{a\}) \cap A = \emptyset$ so A is not connected. If A is infinite and U, V are open non-empty then $A \cap U \cap V \neq \emptyset$ so A is connected. [N, 1]

Let $A = \{a, b\} \times X$. Then $U = X \setminus \{a\}, V = X \setminus \{b\}$ are open in X so $U \times X, V \times X$ are open in $X \times X$, $A \cap (U \times X) \cap (V \times X) = \emptyset$ and $A \subset (U \times X) \cup (V \times X)$. So A is not connected. [N, 2]

Problem 2

(a) i) A family $\mathcal{U} = \{U_i : i \in I\}$ of subsets of a space X is called a cover if $X = \bigcup_{i \in I} U_i$ If each U_i is open in X then $\mathcal{U}$ is called an open cover for X. A subcover of a cover $\{U_i : i \in I\}$ for a space X is a subfamily $\{U_j : j \in J\}$ for some subset $J \subset I$ such that $\{U_j : j \in J\}$ is still a cover for X. We call it a finite subcover if J is finite. A topological space X is *compact* if any open cover of X has a finite subcover. [B, 2]

ii) Let $\{U_i : i \in I\}$ be an open cover of Y. Then $\{f^{-1}(U_i) : i \in I\}$ is an open cover of X. The compactness of X implies that there exists a finite subcover $\{f^{-1}(U_j) : j \in J\}$ Since f is onto, $\{U_j : j \in J\}$ is a finite subcover for Y. [B, 3]

(b) i) Let $X/\mathcal{R}$ be the set of equivalence classes of $\mathcal{R}$. The quotient topology $\mathcal{T}'$ of $X/\mathcal{R}$ consists of the sets U such that $p^{-1}(U)$ is open in X.

Clearly $\emptyset, X/\mathcal{R}$ lie in $\mathcal{T}'$. Let $U_1, U_2 \in \mathcal{T}'$. Then $p^{-1}(U_1 \cap U_2) = p^{-1}(U_1) \cap p^{-1}(U_2)$ so $U_1 \cap U_2 \in \mathcal{T}'$. Also if $\{U_i : i \in I\} \subset \mathcal{T}'$ then

$$p^{-1}(\bigcup U_i) = \bigcup p^{-1}(U_i)$$

so $\bigcup U_i$ lies in $\mathcal{T}'$. [B, 3]

ii) Suppose that g is continuous. By the definition of the quotient topology p is continuous. So, $g \circ p$ is continuous.

Suppose that $g \circ p$ is continuous. Let U be an open subset of Z. Then, by assumption, $g \circ p^{-1}(U)$ is open in X. This is $p^{-1}1(g^{-1}(U))$. By the definition of the quotient topology, $g^{-1}(U)$ is therefore open in $X/\mathcal{R}$. So, g is continuous. [B, 2]

(c) i) We note that p is injective on K. So $p^{-1}(p(X-K)) = X - K$. It follows that $p(X-K)$ is open. [S, 2]

ii) Let $f : [0,1] \to S^1$ where $f(t) = e^{2\pi i t}$. Since $f(0) = f(1)$, this gives a well-defined function $g : X_A \to S^1$ where $g([x]) = f(x)$. So $f = g \circ p$ and g is continuous by (b,ii). Since $X = [0,1]$ is compact X_A is compact as well. As S^1 is Hausdorff and g $1-1$, g is a homeomorphism. [S, 2]

iii) Let $f : [0,1] \to X_A$ where $f(x) = [x]$. Then f is the restriction of p to $[0,1]$ so it is continuous. As $[0,1]$ is compact and f is onto X_A is compact. Let $h : [0,1] \to S^1$ where $h(t) = e^{2\pi i t}$. This gives a well-defined 1-1 and onto function $g : X_A \to S^1$ where $g([x]) = h(x)$ if $x \notin A$ and $g([x]) = h(0) = h(1)$ if $x \in A$. So h is the restriction of $g \circ p$ to $[0,1]$ and g is continuous by (b,ii). Since X_A is compact and S^1 is Hausdorff g is a homeomorphism. [S,4]

iv) Let $[a], [b] \in X_{\mathbb{Z}}$. If $a, b \notin \mathbb{Z}$ pick open neighborhoods U, V of a, b in $\mathbb{R}$ such that $U \cap V = \emptyset$ and $U \cap \mathbb{Z} = \emptyset, V \cap \mathbb{Z} = \emptyset$. Then by i) $p(U), p(V)$ are disjoint open containing respectively $[a], [b]$. If, say $a \in \mathbb{Z}$, let $m = \min\{|n - b|; b \in \mathbb{Z}\}$. If $\epsilon = m/3$ then the sets $U = \bigcup_n (n-\epsilon, n+\epsilon)$, $V = (b-\epsilon, b+\epsilon)$ are an open saturated sets $[a] \in p(U), [b] \in p(V)$ and $p(U) \cap p(V) = \emptyset$. So $X_{\mathbb{Z}}$ is Hausdorff.

Let a, b be distinct irrational numbers. Then if U, V are open sets containing $[a], [b]$ respectively, $p^{-1}(U), p^{-1}(V)$ both intersect $\mathbb{Q}$. It follows that $p(U) \cap p(V) \neq \emptyset$ so $X_{\mathbb{Q}}$ is not Hausdorff. [S, 4]

v) Let S be a countable subset of $\mathbb{R}$. For each $n \in \mathbb{N}$ we pick $x_n \in (n, n+1)$ with $x_n \notin S$. The sets $K_r = \{x_n : n \in \mathbb{N},\, n \geq r\}$ are all closed disjoint from S. So by (c, i) $p(X - K_r)$ is open in X_S. Clearly $X_S = \bigcup_r p(X - K_r)$ and this open cover has no finite subcover, so X_S is not compact. [N, 3]

Problem 3

(a) i) The standard n-simplex is the set

$$\Delta^n = \{(x_1, ..., x_{n+1}) \in \mathbb{R}^{n+1} : x_i \geq 0 \, \forall i \text{ and } \sum_i x_i = 1\}.$$

An abstract simplicial complex is a pair (V, Σ), where V is a set (whose elements are called vertices) and Σ is a set of non-empty finite subsets of V (called simplices) such that

(1) for each $v \in V$, the 1-element set $\{v\}$ is in Σ;

(2) if σ is an element of Σ, so is any non-empty subset of σ.

A face inclusion of a standard m-simplex Δ^m into a standard n-simplex Δ^n (where $m < n$) is a function $\Delta^m \to \Delta^n$ that is the restriction of an injective linear map $\mathbb{R}^{m+1} \to \mathbb{R}^{n+1}$ which sends the vertices of Δ^m to vertices of Δ^n.

The topological realisation $|K|$ of an abstract simplicial complex $K = (V, \Sigma)$ is the space obtained by the following procedure: (1) For each $\sigma \in \Sigma$, take a copy of the standard n-simplex, where $n+1$ is the number of elements of σ. Denote this simplex by Δ_σ. Label its vertices with the elements of σ. (2) Whenever $\sigma \subset \tau \in \Sigma$, identify Δ_σ with a subset of Δ_τ via the face inclusion which sends the elements of σ to the corresponding elements of τ.

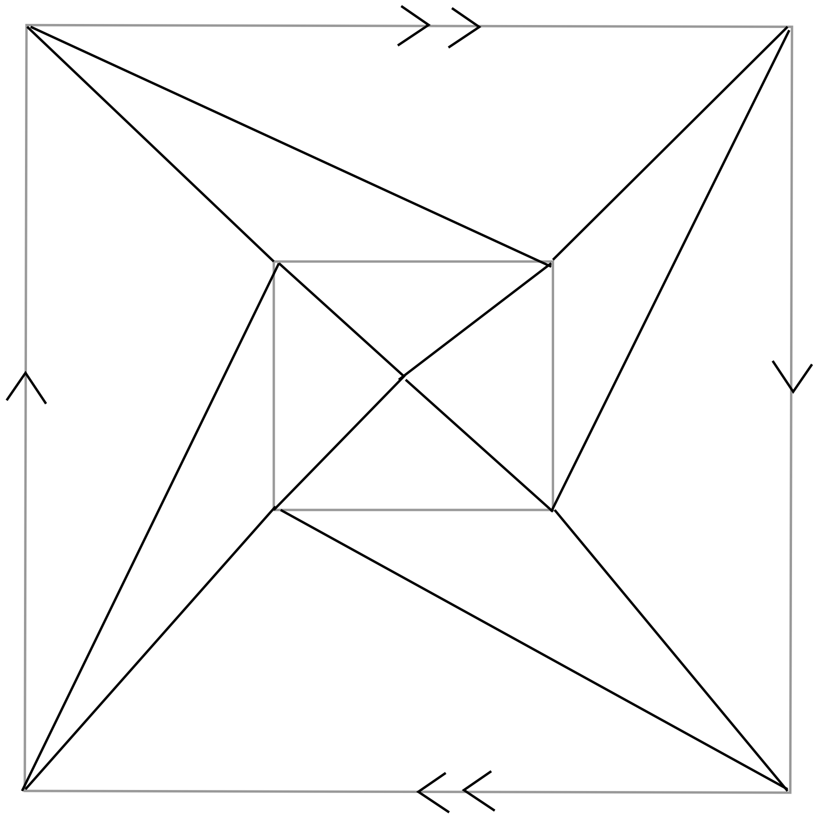
In other words, $|K|$ is a quotient space, obtained by starting with the disjoint union of the simplices in (1), and then imposing the equivalence relation that is described in (2). [B, 4]

ii) Let K' be the simplicial complex with the vertex set V, but where every non-empty subset of V is a simplex. Then $|K'|$is homeomorphic to a standard simplex. The inclusion $|K| \to |K'|$ is a continuous injection. The standard simplex is a subset of $\mathbb{R}^n$. Hence, we obtain the required continuous injection $f : |K| \to \mathbb{R}^n$. Let $a, b \in |K|$. If U, V are disjoint open neighborhoods of $f(a), f(b)$, $f^{-1}(U), f^{-1}(V)$ are disjoint open neighborhoods of a, b. [B and S, 3]

iii) An n-dimensional manifold is a Hausdorff topological space M such that every point of M lies in an open set that is homeomorphic to an open set in $\mathbb{R}^n$. A 2-dimensional manifold is a surface. [B, 2]

iv)

[S, 4]



b) i) Let P be a finite-sided convex polygon in $\mathbb{R}^2$. Suppose that P has an even number of sides. Arrange these sides into pairs. For each such pair, the two sides of the pair will be identified. More specifically, suppose that e and e' are two sides of a pair. If e is an edge of P_1 edge joining (x_1, y_1) to (x_2, y_2) and e' is an edge of P joining some points (x'_1, y'_1) to (x'_2, y'_2) we identify the point $(1-t)(x_1, y_1) + t(x_2, y_2)$ to the point $(1-t)(x'_1, y'_1) + t(x'_2, y'_2)$ where $t \in [0, 1]$.

Once one has chosen the edges e and e' to identify, there is still some choice about how this identification is made, because we can choose e to run from (x_1, y_1) to (x_2, y_2) , or the other way round. We encode this choice by drawing an arrow on e, running from (x_1, y_1) to (x_2, y_2). When arrows have been drawn on both e and e' , this determines how they are identified. This construction is a polygon with a complete set of side identifications. [B, 3]

ii) It is convenient to describe a polygon with side identifications using a word, which is a string of letters, possibly with $()^{-1}$ signs. We start with some vertex of the polygon, and run around the boundary of the polygon. Each pair of edges that is to be identified is given a letter. We orient the edge in some way. When we come to that edge, we write down the letter or its inverse, depending on whether we traverse the edge in the forwards or backwards direction.

With this notation let M_0 be the surface corresponding to the word $xx^{-1}yy^{-1}$. For $g \geq 1$ let M_g be the surface obtained from the word:

$x_1y_1x_1^{-1}y_1^{-1}....x_gy_gx_g^{-1}y_g^{-1}$. Let N_1 be the surface obtained from the word $xxyy^{-1}$. For $h > 1$,let N_h be the surface obtained from the word: $x_1x_1....x_hx_h$.

The classification theorem of closed combinatorial surfaces states that every closed combinatorial surface is homeomorphic to one of the manifolds M_g , for some $g \geq 0$, or N_h , for some $h \geq 1$.[B, 4]

iii) All these squares can be encoded by words. Without loss of generality we may assume that the 1st letter of the word is b. Then the second letter will be either $b^{\pm 1}$ or $c^{\pm 1}$. By relabelling we may assume that if the second letter is not $b^{\pm 1}$ then it is c. Similarly if the second letter is $b^{\pm 1}$ we may assume that the 3rd letter is c.

So, up to relabelling the 4 sides we obtain the following words corresponding to the indicated surfaces:

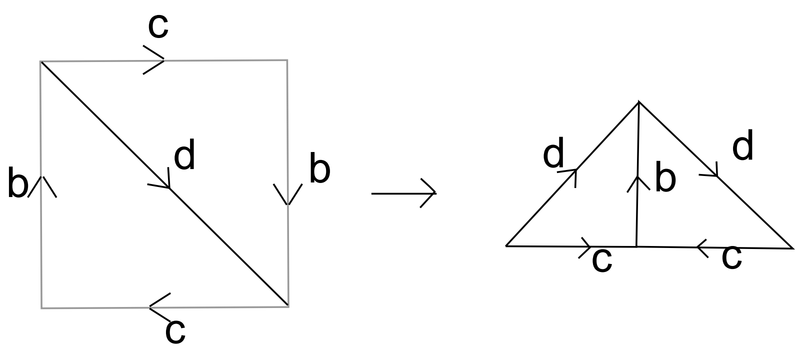
$$bbcc = N_2, bbcc^{-1} = N_1, bb^{-1}cc^{-1} = M_0, bb^{-1}cc = N_1, bcbc = N_1$$

,

$$bcbc^{-1} = N_2, bcb^{-1}c^{-1} = M_1, bcb^{-1}c = N_2$$

We see that the words $bcbc$ corresponds to N_1 and $bcb^{-1}c$ to N_2 by the following 'cut and paste' diagrams:

[N, 5]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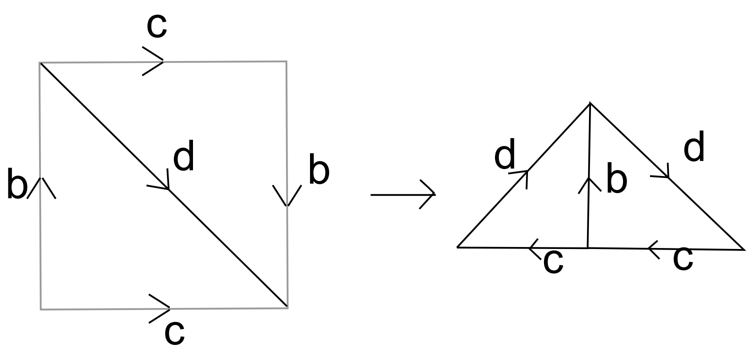
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