1. (a) [8 marks] (i) [2 marks] [**B**] A is connected if and only if, whenever U and V are open subsets of X such that $A \subseteq (U \cup V)$ and $U \cap V \cap A = \emptyset$, either $U \cap A = \emptyset$ or $V \cap A = \emptyset$. Equivalently, any continuous map from A (with the induced topology) to $\{0, 1\}$ (with the discrete topology) is constant.

(ii) [2 marks] [**B**] Let $f : \bigcup_{i \in I} A_i \to \{0, 1\}$ be a continuous map. Its restriction to A_i, $f|_{A_i}$, is also continuous. Since A_i is connected it follows that $f|_{A_i}$ is constant equal to $c_i \in \{0, 1\}$.

(iii) [1 mark] [**B**] Let $g : f(A) \to \{0, 1\}$ be a continuous map. Then $g \circ f|_A : A \to \{0, 1\}$ is a continuous map, and it must be constant, by the connectedness if A.

For each pair $i, j \in I$ there exists a point x in $A_i \cap A_j$, and $f(x) = c_i = c_j$. Therefore all the constants c_i are equal, and f is constant.

(iv) [3 marks] [**B**] A *path connecting two points* x, y in a topological space X is a continuous map $\mathfrak{p} : [0, 1] \to X$ with $\mathfrak{p}(0) = x$, $\mathfrak{p}(1) = y$.
A space is *path-connected* if every two points in it can be connected by a path.

Let A be a path-connected space and let a be a fixed point in A. For every $x \in A$ there exists a path $\mathfrak{p}_x : [0, 1] \to X$ with image in A such that $\mathfrak{p}_x(0) = a$ and $\mathfrak{p}_x(1) = x$. The set $\mathcal{P}_x = \mathfrak{p}_x[0, 1]$ is a connected set, because it is the image of a connected set by a continuous map.

We apply the previous question to the collection of connected sets $\{\mathcal{P}_x : x \in A\}$, and conclude that $\bigcup_{x \in A} \mathcal{P}_x = A$ is connected.

(b) [7 marks] (i) [2 marks] [**S**] Let $C_a = \bigcup_{a \in K, K \text{connected}} K$. This set is connected, by the previous question, contains a, and every connected set containing a.

(ii) [2 marks] [**S**] It suffices to prove that two connected components are either disjoint or coincide. If C_a and C_b are not disjoint, then $C_a \cup C_b$ is connected, and since it contains a, respectively b, it must equal C_a, respectively C_b.

(iii) [3 marks] [**N**] Clearly $A \subset \bigcup_{a \in A} C_a$. For every $a \in A$, $C_a \subset A \cup (X \setminus A)$, both A and $X \setminus A$ are open, they are disjoint, and $A \cap C_a \neq \emptyset$, hence $(X \setminus A) \cap C_a = \emptyset$. It follows that $C_a \subseteq A$. Therefore $\bigcup_{a \in A} C_a \subset A$, whence $\bigcup_{a \in A} C_a = A$.

(c) [10 marks] (i) [2 marks] [**B**] The closure can be defined as

$$\bar{A} = \{x \in X \mid \text{ for every open set } U \text{ containing } x, U \cap A \neq \emptyset\}.$$

From the above definition it is clear that $A \subset \bar{A}$, therefore an open set intersecting A intersects $\bar{A}$. Conversely, if an open set U contains a point $x \in \bar{A}$ then by the definition of $\bar{A}$, U intersects A.

(ii) [5 marks] [**B**, **N**] The closure of a connected set is connected. Indeed if $\{U, V\}$ is an open cover of $\bar{A}$ such that $U \cap V \cap \bar{A} = \emptyset$ then the same thing is true about A. As A is connected, either $U \cap A = \emptyset$ or $V \cap A = \emptyset$. This and the previous question imply that either $U \cap \bar{A} = \emptyset$ or $V \cap \bar{A} = \emptyset$.
From the previous question it follows that if A is path-connected then $\bar{A}$ is connected. It is nevertheless not true that $\bar{A}$ is path-connected. This can be seen for instance by taking A to be the graph $\{(x, \sin 1/x) \mid x \in (0, 1/\pi]\}$, path-connected, therefore connected. Its closure $\bar{A}$ is equal to $A \cup [\{0\} \times [-1, 1]]$. It is not path-connected, as seen in the Metric Spaces course and recalled in the Topology course.

(iii) [3 marks] [**N**] The closure of every connected component of A is contained in a connected component of $\bar{A}$.

The inclusion is nevertheless strict: several connected components of A can be contained in the same connected component of $\bar{A}$. Example: $A = \mathbb{Q}$. Its connected components are points while $\bar{A} = \mathbb{R}$ has only one connected component. The same example also shows that a connected component of $\bar{A}$ can be strictly larger than the union of the closures of all the connected components contained in it.

2. (a) [6 marks] Let $(X, \mathcal{T})$ be a topological space.
   (i) [2 marks] [**B**] $f : X \to Y$ is *continuous* if for every U open in Y, $f^{-1}(U)$ is open in X.
   (ii) [4 marks] [**S**] $V = \left(\frac{f(x_0)}{2}, \infty\right)$ is open in $\mathbb{R}$ with the standard topology and f continuous, therefore $U = f^{-1}(V)$ is open and contains x_0.

   (b) [12 marks] (i) [4 marks] [**B**] (a) A subset A of X is *compact* if any open cover of A has a finite subcover.

   (b) Equivalently, if $\{V_i : i \in I\}$ is an indexed family of closed subsets of X such that $\bigcap_{j \in J} V_j \cap A \neq \emptyset$ for any finite subset $J \subseteq I$ then $\bigcap_{i \in I} V_i \cap A \neq \emptyset$.

   (a) $\Rightarrow$ (b) Assume that $\{V_i : i \in I\}$ is a family of closed subsets of X such that $\bigcap_{j \in J} V_j \cap A \neq \emptyset$ for any finite subset $J \subseteq I$, while $\bigcap_{i \in I} V_i \cap A = \emptyset$. Then $\bigcup_{i \in I}(X \setminus V_i)$ is an open cover of A.

   According to (a) there exists J finite subset in I such that $A \subset \bigcup_{j \in J}(X \setminus V_j)$. This is equivalent to $\bigcap_{j \in J} V_j \cap A = \emptyset$, contradicting the hypothesis.

   (b) $\Rightarrow$ (a) Let $\{U_i : i \in I\}$ be an open cover of A. Then $A \subset \bigcup_{i \in I} U_i$, whence $\bigcap_{i \in I}(X \setminus U_i) \cap A = \emptyset$.

   According to property (b) applied to the family of closed sets $\{X \setminus U_i : i \in I\}$ there exists some finite subset J of I such that $\bigcap_{j \in J}(X \setminus U_j) \cap A = \emptyset$. Equivalently $\bigcup_{j \in J} U_j$ contains A.

   (ii) [5 marks] [**B**] For every $y \in K$, according to the Hausdorff property, there exists a pair of disjoint open sets U_y and V_y, $y \in U_y$, $x \in V_y$. The collection $\{U_y : y \in K\}$ is an open cover for K. Therefore there exist $y_1, ..., y_m$ such that $K \subseteq U_{y_1} \cup \cdots \cup U_{y_m}$. Let $U = U_{y_1} \cup \cdots \cup U_{y_m}$.

   The set $V = V_{y_1} \cap \cdots \cap V_{y_m}$ is open, it contains x, and $V \cap U = V \cap \bigcup_{j=1}^{m} U_{y_j} = \emptyset$. Thus, for every $x \in X \setminus K$, there exists V open such that $x \in V \subseteq X \setminus K$. This implies that $X \setminus K$ is open.

   (iii) [3 marks] [**N**] According to the second question of the exercise, every $x \in X$ is contained in an open set U_x such that for every $y \in U_x$, $f(y) > f(x)/2$. Since $\bigcup_{x \in X} U_x$ is an open cover of X and X is compact, there exists an open subcover $U_{x_1} \cup U_{x_2} \cup \cdots \cup U_{x_n}$. We can take $c = \inf_{i \in \{1,\ldots,n\}} \frac{f(x_i)}{2}$.

   (c) [7 marks] (i) [2 marks] [**N**] The sum $S = f_1 + f_2 + \cdots + f_n$ is in $\mathcal{F}$, therefore there exists a point x_A such that $S(x_A) = 0$. This and the fact that $f_i(x_A) \geqslant 0$ for every $i \in \{1, \ldots, n\}$ implies that $f_i(x_A) = 0$ for every $i \in \{1, \ldots, n\}$.

   (ii) [5 marks] [**N**] The closed subsets $V_f = \{x \in X \mid f(x) = 0\}$ have the property that every finite collection has a non-empty intersection. This and the fact that X is compact implies that $\bigcap_{f \in \mathcal{F}} V_f$ is non-empty.

3. (a) [6 marks] Let $(X, \mathcal{T})$ be a topological space.

(i) [2 marks] [**B**] A subset A of X is *saturated* (with respect to $\mathcal{R}$) if it is a union of equivalence classes.

(ii) [2 marks] [**B**] Let $p\colon X \to X/\mathcal{R}$ be the map that assigns to each point x in X the equivalence class $[x]$, called the *collapsing map* in lectures.
A subsets $\tilde{U}$ of $X/\mathcal{R}$ is open in the quotient topology if and only if $p^{-1}(\tilde{U}) \in \mathcal{T}$.

(iii) [2 marks] [**B**] A quotient space $X/\mathcal{R}$ with the quotient topology is Hausdorff if and only if any two distinct equivalence classes are contained in two disjoint open saturated sets.

(b) [13 marks] (i) [7 marks] [3 marks][**S**] Let X be $\mathbb{R}$ with the standard metric. The quotient space $X/\mathbb{Z}$ is Hausdorff. Indeed consider two distinct equivalence classes in it, $a+\mathbb{Z}$ and $b+\mathbb{Z}$. Wlog we may assume that $a, b \in [0,1)$. The two classes are contained in the disjoint saturated open sets $(a-\varepsilon, a+\varepsilon)+\mathbb{Z}$ and $(b-\varepsilon, b+\varepsilon)+\mathbb{Z}$, where $\varepsilon \leqslant \frac{|a-b|}{2}$.
[4 marks][**S**] The quotient space $X/\mathbb{Q}$ is not Hausdorff. Indeed, the quotient topology of this space is the indiscrete topology. Let $p : \mathbb{R} \to \mathbb{R}/\mathbb{Q}$ be the quotient map $p(x) = x + \mathbb{Q}$. Given a non-empty open set U in $\mathbb{R}/\mathbb{Q}$, $V = p^{-1}(U)$ is open in $\mathbb{R}$ and for every element a in it, it contains $a + \mathbb{Q}$. Given an arbitrary number x, $-x + V$ is open, hence it contains an open interval, hence it intersects $\mathbb{Q}$. Therefore V intersects $x + \mathbb{Q}$, hence it contains it. It follows that U is the entire space $\mathbb{R}/\mathbb{Q}$.

(ii) [6 marks] [1 mark][**S**] The three properties of an equivalence relation are satisfied by $\mathcal{R}$ because they are satisfied by the equality relation.

[5 marks][**N**] Consider two distinct equivalence classes $[x] \neq [y]$. Then there exists $f : X \to Y$ continuous, with Y Hausdorff, such that $f(x) \neq f(y)$. It follows that there exist V_x, V_y open disjoint sets in Y such that $f(x) \in V_x$ and $f(y) \in V_y$. Then the open subsets $U_x = f^{-1}(V_x)$ and $U_y = f^{-1}(V_y)$ are open disjoint subsets of X containing x, respectively y.
Moreover, both U_x and U_y are saturated. Indeed, let $[z]$ be an equivalence class intersecting U_x. Then $f(z) \in V_x$. For every $w\mathcal{R}z$, by the definition of $\mathcal{R}$ it follows that $f(w) = f(z) \in V_x$, whence $w \in U_x = f^{-1}(V_x)$.

(c) [6 marks] (i) [2 marks] [**S**] The fact that $\mathcal{T}$ is a topology follows from the fact that finite unions and arbitrary intersections of finite sets are finite.

(ii) [4 marks] [**N**] Let $f : X \to Y$ be a continuous map to a Hausdorff space Y. Assume that there exist $a, b \in X$ such that $f(a) \neq f(b)$. It follows that there exist U, V open disjoint subsets in Y containing $f(a)$, respectively $f(b)$. Then $U' = f^{-1}(U)$ and $V' = f^{-1}(V)$ are open non-empty disjoint subsets in X. By definition $U' = X \setminus F_1$ and $V' = X \setminus F_2$, where F_1 and F_2 are finite. We have that $\emptyset = U' \cap V' = X \setminus (F_1 \cup F_2)$, whence $X = F_1 \cup F_2$, contradicting the fact that X is infinite.
Thus every continuous map $f : X \to Y$ to a Hausdorff space Y is constant. It follows that every two points in X are equivalent, hence $X/\mathcal{R}$ is a singleton with the indiscrete topology.