Solutions Topology A.5, 2018-19

Problem 1

a i) A family $\mathcal{U} = \{U_i : i \in I\}$ of subsets of a space X is called a cover if $X = \bigcup_{i \in I} U_i$. If each U_i is open in X then $\mathcal{U}$ is called an open cover for X. A subcover of a cover $\{U_i : i \in I\}$ for a space X is a subfamily $\{U_j : j \in J\}$ for some subset $J \subset I$ such that $\{U_j : j \in J\}$ is still a cover for X. We call it a finite subcover if J is finite. A topological space X is *compact* if any open cover of X has a finite subcover. $K \subset X$ is compact if it is a compact topological space with respect to the subspace topology. [B, 2]

A topological space X is path-connected if for any $a, b \in X$ there is $f : [0,1] \to X$ continuous such that $f(0) = a$, $f(1) = b$. [B, 1]

a ii) A basis for the topology of $X \times Y$ is given by $\mathcal{B} = \{U \times V : U \in \mathcal{T}_X, V \in \mathcal{T}_Y\}$. More explicitly $U \subset X \times Y$ is open if it can be written as union of elements of B. [B,1]

If $U \subset X$ is open then $p_X^{-1}(U) = U \times Y$ which is clearly open in $X \times Y$. So p_X is continuous. Similarly p_Y is continuous. If f is continuous $p_X \circ f, p_Y \circ f$ are continuous as compositions of continuous functions. [B, 3]

Assume now the functions $p_X \circ f, p_Y \circ f$ are continuous. Let U be open in X and V be open in Y. Then

$$f^{-1}(U \times V) = (p_X \circ f)^{-1}(U) \cap (p_Y \circ f)^{-1}(V)$$

so $f^{-1}(U \times V)$ is open. Since the sets $U \times V$ form a basis of $X \times Y$ f is continuous. [B, 3]

a iii) Suppose $X \times Y$ is path connected. Let $a, b \in X$ and $c \in Y$. There is a continuous $f : [0,1] \to X \times Y$ continuous with $f(0) = (a, c)$, $f(1) = (b, c)$. Then $p_X \circ f$ is a path joining a, b in X so X is path connected.

Conversely if $(a_1, b_1), (a_2, b_2)$ are points in $X \times Y$ and h, g are paths joining a_1, a_2 in X and b_1, b_2 in Y respectively then $F : [0,1] \to X \times Y$ given by $F(x) = (h(x), g(x)$ is a continuous path joining $(a_1, b_1), (a_2, b_2)$ since h, g are continuous. [S, 3]

b i) The only open sets of Y are $\emptyset, Y$ and $f^{-1}(\emptyset), f^{-1}(Y)$ are open so f is continuous. Define $f : [0,1] \to Y$ by $f(0) = 0, f(x) = 1$ for $x > 0$. Clearly f is a path joining $[0,1]$ so Y is path connected. Since $\mathbb{R}$ is path-connected by part a $\mathbb{R} \times Y$ is path connected. [S, 2]

By contradiction: say $f(x,0)=a, f(x,1)=b$. Then there is an open set U which contains a but does not contain b. Then $f^{-1}(U)$ is open so it is of the form $O\times Y$ with $O\subseteq\mathbb{R}$ open. Since $(x,0)\in O\times Y$ we have $(x,1)\in O\times Y$ so $b\in U$ which is a contradiction. [S, 2]

b ii) A is compact. Let $\mathcal{U}=\{U_i : i\in I\}$ be an open cover of A. Then each U_i is of the form $V_i\times Y$ where V_i is open in $\mathbb{R}$. Since $[a,b]$ is compact there is a finite subcover $\{V_j : j\in J\}$. Then $\{V_j\times Y : j\in J\}$ is a finite subcover of $\mathcal{U}$. [S, 4]

b iii) Since B is bounded $B\subseteq[a,b]$ for some interval in $\mathbb{R}$. Let $C=[a,b]\setminus B$ and

$$K_1=B\times\{0\}\cup C\times\{1\}\ K_2=[a,b]\times\{0\}.$$

As in b ii) K_1,K_2 are compact in $\mathbb{R}\times Y$ and clearly $K_1\cap K_2=B\times\{0\}$. [N, 4]

Problem 2

a i) A topological space X is disconnected if there are disjoint open non-empty subsets U and V such that $U\cup V=X$. If X is not disconnected, it is called connected.

A subset A of X is connected if it is a connected topological space with respect to the subspace topology. [B, 2]

Let U and V be open sets in Y such that $f(A)\subseteq U\cup V$ and $f(A)\cap U\cap V=\emptyset$. Then $f^{-1}(U)$ and $f^{-1}(V)$ are open in X. So, $f^{-1}(U)\cap A$ and $f^{-1}(V)\cap A$ are disjoint and open in A. Since A is connected, one of $f^{-1}(U)\cap A$, $f^{-1}(V)\cap A$ is empty. Hence, one of $U\setminus f(A)$ and $V\setminus f(A)$ is empty. So, $f(A)$ is connected. [B, 4]

a ii) Let $X/\mathcal{R}$ be the set of equivalence classes of $\mathcal{R}$. The quotient topology $\mathcal{T}$ of $X/\mathcal{R}$ consists of the sets U such that $p^{-1}(U)$ is open in X. Clearly $\emptyset, X/\mathcal{R}$ lie in $\mathcal{T}$. Let $U_1,U_2\in\mathcal{T}$. Then $p^{-1}(U_1\cap U_2)=p^{-1}(U_1)\cap p^{-1}(U_2)$ so $U_1\cap U_2\in\mathcal{T}$. Also if $\{U_i : i\in I\}\subset\mathcal{T}$ then

$$p^{-1}(\bigcup U_i)=\bigcup p^{-1}(U_i)$$

so $\bigcup U_i$ lies in $\mathcal{T}$. [B, 4]

b i) It follows by the definition of the quotient topology $p(K)$ is closed in X/C if $p^{-1}(p(K))$ is closed in X. We note that if $K\cap C=\emptyset$, $p^{-1}(p(K))=K$ while if $K\cap C\neq\emptyset$, $p^{-1}(p(K))=K\cup C$, so in both cases $p^{-1}(p(K))$ is closed. [S, 3] b ii) $\mathbb{R}^2/A$ is not Hausdorff since any open set in X/A containing $[(1,0)]$ intersects A by the definition of the quotient topology on X/A. So there do not exist open disjoint U, V in X/A containing $[(1,0)], A$ respectively. [S, 3]

b iii) Clearly $\mathbb{R}^2/Y \setminus \{Y\}$ is not connected. However $\mathbb{R}^2 \setminus \{q\}$ is connected (as it is path-connected) for any $q \in \mathbb{R}^2$. Therefore $\mathbb{R}^2/Y$ and $\mathbb{R}^2$ are not homeomorphic. [S, 3]

Let U be an open set in $\mathbb{R}^2/Y$ containing Y. Then $p^{-1}(U)$ contains $(1/n, 1)$ for sufficiently large n. So $[(1/n, 1)]$ lies in U for sufficiently large n and $[(1/n, 1)]$ converges to Y. [N, 2]

$\{(1/n, n) : n \in \mathbb{N}\}$ is a closed subset of $\mathbb{R}^2$ that does not intersect Y. Therefore by b i) $\{[(1/n, n)] : n \in \mathbb{N}\}$ is a closed subset of $\mathbb{R}^2/Y$ that does not intersect Y, so it does not converge to Y. [N, 4]

Problem 3

(a) i) An abstract simplicial complex is a pair (V, Σ), where V is a set (whose elements are called vertices) and Σ is a set of non-empty finite subsets of V (called simplices) such that

(1) for each $v \in V$, the 1-element set $\{v\}$ is in Σ;

(2) if σ is an element of Σ, so is any non-empty subset of σ.

The standard n-simplex is the set

$$\Delta^n = \{(x_1, ..., x_{n+1}) \in \mathbb{R}^{n+1} : x_i \geq 0 \, \forall i \text{ and } \sum_i x_i = 1\}.$$

A face inclusion of a standard m-simplex Δ^m into a standard n-simplex Δ^n (where $m < n$) is a function $\Delta^m \to \Delta^n$ that is the restriction of an injective linear map $\mathbb{R}^{m+1} \to \mathbb{R}^{n+1}$ which sends the vertices of Δ^m to vertices of Δ^n.

The topological realisation $|K|$ of an abstract simplicial complex $K = (V, \Sigma)$ is the space obtained by the following procedure: (1) For each $\sigma \in \Sigma$, take a copy of the standard n-simplex, where $n+1$ is the number of elements of σ. Denote this simplex by Δ_σ. Label its vertices with the elements of σ. (2) Whenever $\sigma \subset \tau \in \Sigma$, identify Δ_σ with a subset of Δ_τ via the face inclusion which sends the elements of σ to the corresponding elements of τ.

In other words, $|K|$ is a quotient space, obtained by starting with the disjoint union of the simplices in (1), and then imposing the equivalence relation that is described in (2).

The link of v , denoted $lk(v)$, is the subcomplex with vertex set $\{w \in V \setminus \{v\} : \{v, w\} \in \Sigma\}$ and with simplices σ such that $v \notin \sigma$ and $\{v\} \cup \sigma \in \Sigma$. [B, 5]

a ii) The realisation $|K|$ of a simplicial complex K is obtained from a disjoint union of simplices, by forming a quotient space. The disjoint union of finitely many simplices is compact. Hence, the quotient space is the continuous image of a compact space and therefore compact.

Let K' be the simplicial complex with the vertex set V , but where every non-empty subset of V is a simplex. Then $|K'|$ is homeomorphic to a standard simplex. The inclusion $|K| \to |K'|$ is a continuous injection. The standard simplex is a subset of $\mathbb{R}^n$. Hence, we obtain a continuous injection $i : |K| \to \mathbb{R}^n$. If $a, b \in |K|$ are distinct there are U, V open disjoint set containing $i(a), i(b)$ respectively. Then $i^{-1}(U), i^{-1}(V)$ are disjoint open sets containing a, b respectively. [B, 5]

a iii) If K contains simplices of dimension 2 then there are points a, b lying on a 2-simplex such that $|K| \setminus \{a, b\}$ is connected. It follows that K is of dimension 1. $|K|$ is also connected. Suppose that the link of some vertex v contains at least 3 points. We note that $|K| \setminus \{v\}$ is connected. Let e_1, e_2, e_3 be three 1-simplices adjacent to v with vertices $\{v, v_1\}$, $\{v, v_2\}$, $\{v, v_3\}$ respectively.

Since $|K| \setminus \{v\}$ is path connected there is a path joining v_i, v_j in $|K| \setminus \{v\}$ for $i, j \in \{1, 2, 3\}$. Therefore, without loss of generality we may assume that there is a path in $|K| \setminus \{v\}$ from v_3 to v_1 which does not contain v_2 and a path from v_3 to v_2 which does not contain v_1. Let u_1, u_2 be points in the interior of e_1, e_2 respectively.

Then $|K| \setminus \{u_1, u_2\}$ is connected contradicting our assumption that $|K|$ is homeomorphic to S^1. Similarly if the link of v contains only one point, e is a 1-simplex adjacent to v and w is a vertex in its interior then $|K| \setminus \{v\}$ is not connected. We conclude that $lk(v)$ has two points for every vertex v. [N, 4]

b) A surface is a Hausdorff topological space S such that every point of S lies in an open set that is homeomorphic to an open set in $\mathbb{R}^2$. [B, 1]

A *closed combinatorial surface* is a connected finite simplicial complex K such that for every vertex v of K, the link of v is a simplicial circle. [B, 1]

Note that $|K|$ is Hausdorff by a ii). Each point of $|K|$ lies in the inside of a 2-simplex, a 1-simplex or a 0-simplex. In each case, it has a neighbourhood homeomorphic to an open disc in $\mathbb{R}^2$. For a point in the inside of a 2-simplex, this is clear. Let $\{v_1, v_2\}$ be a 1-simplex. Then v_2 lies in the link of v_1. This link is a simplicial circle. So, there are precisely two 1-simplices in this link that are adjacent to v_2. Hence, there are precisely two 2-simplices in $|K|$ that have $\{v_1, v_2\}$ as a face. It follows that any point in the interior of a 1-simplex has a neighbourhood homeomorphic to an open disc.

For a vertex, its link is a simplicial circle, and so its star is homeomorphic to an open disc. [B, 4]

If $|lk(v)|$ is homeomorphic to $[0, 1]$ then for each vertex x in $lk(v)$, $lk(x)$ has at most 2 points. Indeed if not then either $|lk(v)| \setminus \{x\}$ has at least 3 connected components or $lk(v)$ contains a simplicial circle. However in the latter case for any point y in the interior of a 1-simplex on the simplicial circle $|lk(v)| \setminus \{y\}$ is connected. Since $lk(v)$ is not a circle for at least one vertex $lk(x)$ has one point. As K is finite there are exactly 2 vertices of $lk(v)$ such that their link contains exactly 1 point.

Let v be a vertex of K with $|lk(v)|$ homeomorphic to $[0, 1]$. Let x, y be the two vertices of $lk(v)$ which are adjacent to exactly one vertex of $lk(v)$. Then $\{v, x\}, \{v, y\}$ are 1-simplices that lie in exactly one 2-simplex and for each x, y we have that $|lk(x)|$, $|lk(y)|$ is homeomorphic to $[0, 1]$. Continuing the same way we see that v lies in a simplicial cicrle and for each vertex z of this circle $|lk(z)|$ is homeomorphic to $[0, 1]$.

Arguing the same way for each vertex w with $|lk(w)|$ homeomorphic to $[0, 1]$ we see that all such vertices lie in a disjoint union of simplicial circles. Identifying the boundaries of appropriately subdivided discs along these circles we obtain a combinatorial surface S. So $|K|$ is homeomorphic to $|S|$ minus finitely many open disks. [N, 5]