1

(a) $x^2 y'' + 2xy' + y = 0$

$y = x^m \longrightarrow m(m-1) + 2m + 1 = 0$

$\boxed{3}$

$\Rightarrow m^2 + m + 1 = 0 \Rightarrow m = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

$\Rightarrow$ gen. sol'n is $y = x^{-\frac{1}{2}}\left(A\cos\left(\frac{\sqrt{3}}{2}\ln x\right) + B\sin\left(\frac{\sqrt{3}}{2}\ln x\right)\right)$

$\boxed{3}$

(b) $Ly \equiv y'''' + y$

$\langle Ly, w\rangle = \int_0^1 (y'''' + y) w \, dx$ // int. by parts 4 times $\boxed{3}$

$= \underbrace{\left(y'''w - y''w' + y'w'' - yw'''\right)\Big|_0^1}_{\text{must } = 0} + \int_0^1 \underbrace{(w'''' + w)}_{L^*w \,=\, Lw} y \, dx$

$\boxed{3}$ $\boxed{1}$

$= y'''(1)w(1) - y'''(0)w(0) - y''(1)w'(1) + y''(0)w'(0)$

(since $y = y' = 0$ at $0, 1$)

$\Rightarrow$ BC for adjoint: $w(0) = w(1) = w'(0) = w'(1) = 0$

It is fully self-adjoint $\boxed{2}$

c)

$$Ly \equiv y'' \quad , \quad B_1 y \equiv y(1) + y'(2) = 0$$
$$B_2 y \equiv y(2) + y'(1) = 0$$

The GF $g(x,t)$ should satisfy

$$Lg = \delta(x-t) \qquad (1)$$

$\boxed{1}$

and $B_1 g = B_2 g = 0 \qquad (2)$

Letting $g(x,t) = A_1 y_1(x) + A_2 y_2(x) + \tilde{g}(x,t)$

w/ $Ly_i = 0 \; , \; B_i y_i = 0 \quad , \quad i = 1, 2$

(1) $\Rightarrow$ need $L\tilde{g} = \delta(x-t)$

and choose A_1, A_2 to satisfy (2)

$Ly = 0 \Rightarrow y = Ax + B$

$y_1 = \tilde{A}_1 x + \tilde{B}_1 \; , \; B_1 y_1 = 0 \Rightarrow 2\tilde{A}_1 + \tilde{B}_1 = 0$

$\boxed{2}$

so can take $y_1 = x - 2$

$y_2 = \tilde{A}_2 x + \tilde{B}_2 \; , \; B_2 y_2 = 0 \Rightarrow 3\tilde{A}_2 + \tilde{B}_2 = 0$

$\Rightarrow$ take $y_2 = x - 3$

$L\tilde{g} = 0$ in $x<t$, $x>t$ plus $\tilde{g}\Big|_{t^-}^{t^+} = 0$ (3), $\tilde{g}'\Big|_{t^-}^{t^+} = 1$ (4)

$$\Rightarrow \tilde{g} = \begin{cases} Ax+B & x<t \\ Cx+D & x>t \end{cases}$$

$\boxed{3}$

(3) $\to$ $Ct+D-At-B=0$

(4) $\to$ $C-A=1 \Rightarrow C=A+1 \Rightarrow D=B-t$

$$\Rightarrow \tilde{g} = \begin{cases} Ax+B & 1<x<t \\ (A+1)x+B-t & t<x<2 \end{cases}$$

A, B are arb., can set $A=B=1$ WLOG, $\to \tilde{g} = \begin{cases} x+1 & x<t \\ 2x+1-t & x>t \end{cases}$

Now, $B_1 g = A_1 \cancel{B_1 y_1}^{\,0} + A_2 B_1 y_2 + B_1\tilde{g} = 0$

$$\Rightarrow A_2 = \frac{-B_1\tilde{g}}{B_1 y_2}.$$

$\boxed{2}$

Similarly, $A_1 = \dfrac{-B_2\tilde{g}}{B_2 y_1}$

$B_1\tilde{g} = 2+2 = 4$, $B_2\tilde{g} = 4+1-t+1 = 6-t$

$B_1 y_2 = 1-3+1 = -1$, $B_2 y_1 = 2-2+1 = 1$

$\boxed{2}$

$\Rightarrow A_1 = t-6$, $A_2 = 4$

Soln to $Ly=f$ is $y = \int_1^2 g(x,t)\, f(t)\, dt$ w/

$$g(x,t) = (t-6)(x-2) + 4(x-3) + \begin{cases} x+1 & x<t \\ 2x+1-t & x>t \end{cases}$$

2

(a)

$$2xy'' + (x^2-1)y' + \lambda x y = 0$$

ie $$y'' + \underbrace{\frac{x^2-1}{2x}}_{= P(x)} y' + \underbrace{\frac{\lambda x}{2x}}_{= Q(x)} y = 0$$

$x=0$ regular sing pt b/c P is singular at $x=0$, but $x \cdot P(x) = \frac{x^2-1}{2}$ is analytic at $x=0$ & so is $x^2 Q(x) = x^2 \frac{\lambda}{2}$

2

$x=1$ is ordinary point, as neither P nor Q are singular as $x \to 1$

1

$x=\infty$ Define $t = \frac{1}{x}$, $w(t) = y(x)$. $\frac{d}{dx} = -t^2 \frac{d}{dt}$

2

$$\frac{d^2}{dx^2} = -t^2 \frac{d}{dt}\left(-t^2 \frac{d}{dt}\right) = t^4 \frac{d^2}{dt^2} + 2t^3 \frac{d}{dt}$$

$\Rightarrow$ ODE becomes $$\frac{2}{t}\left(t^4 w'' + 2t^3 w'\right) + \left(\frac{1}{t^2} - 1\right)\left(-t^2 w'\right) + \lambda \cdot \frac{1}{t} w = 0$$

$$\Rightarrow 2t^3 w'' + (5t^2-1)w' + \frac{\lambda}{t} w = 0$$

$t=0$ is irreg. sing. pt, b/c $t^2 \cdot \frac{\lambda}{2t^3 \cdot t}$ (ie $t^2 Q(t)$) is singular at $t=0$, & so is $t \cdot \frac{5t^2-1}{t^3}$

$\therefore$ $x=\infty$ is irregular sing pt.

For $x_i = 0$, to get indicial eqn plug in $y = x^\alpha$ and look for balance at lowest order:

$$\underbrace{\left(2\alpha(\alpha-1) - \alpha\right)}_{\displaystyle F(\alpha)} x^{\alpha-1} + O(x^\alpha) = 0$$

$$F(\alpha) = \alpha(2\alpha - 3) = 0 \Rightarrow \alpha = 0, \tfrac{3}{2}$$

are indicial exponents.

3

(b) Since indicial exp. differ by non-integer, get 2 Frobenius sol'ns

3

$$y_1(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}} = a_0 x^{3/2} + a_1 x^{5/2} + \ldots$$

$$y_2(x) = \sum_{k=0}^{\infty} b_k x^k = b_0 + b_1 x + \ldots$$

WLOG set $a_0 = b_0 = 1$

and gen. sol'n is $y = c_1 y_1(x) + c_2 y_2(x)$

3

$y(0) = 0 \Rightarrow c_2 = 0$ (since $y_2(0) = b_0 \neq 0$)

Thus, $y \sim x^{3/2}$ as $x \to 0$

2

(c)

$$2xy'' + (x^2-1)y' + \lambda xy = 0$$

Looking for a sol'n $y = \sum_{k=0}^{\infty} b_k x^k$, we obtain upon plugging in:

$$0 = \sum_{k=0}^{\infty} \left(2k(k-1) - k\right) b_k x^{k-1} + \sum_{k=0}^{\infty} (k+\lambda) b_k x^{k+1}$$

$(\tilde{k}-1 = k+1)$

$$\Rightarrow 0 = \sum_{k=0}^{\infty} k(2k-3) b_k x^{k-1} + \sum_{\tilde{k}=2}^{\infty} (\tilde{k}-2+\lambda) b_{\tilde{k}-2} x^{\tilde{k}-1}$$

$\boxed{3}$

$\Rightarrow$ recursion relation $b_k = \dfrac{-k+2-\lambda}{k(2k-3)} b_{k-2}$

First few terms are: $0 = -b_1 + 2b_2 x + \dots$

$+ \lambda b_0 x + O(x^2)$

$\Rightarrow b_0 = 1$ WLOG, $b_1 = 0 = b_3 = b_5 = \dots$ $\boxed{3}$

$b_2 = -\frac{\lambda}{2} \cdot b_0$, $b_4 = -\dfrac{2+\lambda}{4(5)} \cdot b_2$, $b_6 = -\dfrac{4+\lambda}{6(9)} \cdot b_4$ ...

Thus, if $\lambda = -2m$ for $m \in \mathbb{N}_0$, we get a polynomial sol'n of degree $N = 2m$

$\boxed{3}$

3

(a) [2] [2]

$$\lim_{\varepsilon\to 0^+} \frac{\varepsilon^2 \ln\varepsilon}{\varepsilon} = \lim_{\varepsilon\to 0^+} \frac{\ln\varepsilon}{\frac{1}{\varepsilon}} \underset{\text{L'Hospitals}}{=} \lim_{\varepsilon\to 0^+} \frac{\frac{1}{\varepsilon}}{-\frac{1}{\varepsilon^2}} = 0$$

Thus, $\varepsilon^2 \ln\varepsilon = o(\varepsilon)$ as $\varepsilon \to 0^+$

(b)

$$x^3 + \varepsilon x + 2\varepsilon^2 = 0$$

① ② ③

First take expansion $x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 \ldots$

$\underline{O(1)}$ $x_0^3 = 0 \Rightarrow x_0 = 0$

$\rightarrow (\varepsilon x_1 + \ldots)^3 + \varepsilon(\varepsilon x_1 + \varepsilon^2 x_2 \ldots) + 2\varepsilon^2 = 0$ [2]

$\underline{O(\varepsilon^2)}$ $x_1 + 2 = 0$, $x_1 = -2$

$\underline{O(\varepsilon^3)}$ $x_1^3 + x_2 = 0 \Rightarrow x_2 = 8$

$\Rightarrow$ root $\boxed{x = -2\varepsilon + 8\varepsilon^2 + O(\varepsilon^3)}$

Consider dominant balance

① ~ ③ $x \sim \varepsilon^{2/3} \rightarrow \varepsilon x \sim \varepsilon^{5/3} \gg \varepsilon^2$ ✗ [2]

① ~ ② $x \sim \varepsilon^{1/2}$, $x^3 \rightarrow \varepsilon^{3/2} \gg \varepsilon^2$ ✓

[Note ② ~ ③ is the root just found]

Scale $y = \frac{x}{\varepsilon^{1/2}} \rightarrow \varepsilon^{3/2} y^3 + \varepsilon^{3/2} y + 2\varepsilon^2 = 0$

$\Rightarrow y^3 + y + 2\varepsilon^{1/2} = 0$ $\boxed{1}$

Expand $y \sim y_0 + \varepsilon^{1/2} y_1 + \varepsilon y_2 \cdots$

<u>$O(1)$</u> $y_0(y_0^2 + 1) = 0 \Rightarrow y_0 = 0, \pm i$

<u>$O(\varepsilon^{1/2})$</u> $3y_0^2 y_1 + y_1 + 2 = 0$, $y_0 = 0 \rightarrow y_1 = -2$ is the root already found

$y_0 = \pm i \rightarrow -2y_1 + 2 = 0$

$\rightarrow y_1 = 1$ $\boxed{2}$

$\Rightarrow$ roots $y = \pm i + \varepsilon^{1/2} + O(\varepsilon)$

$$\rightarrow \boxed{x = \pm i \varepsilon^{1/2} + \varepsilon + O(\varepsilon^{3/2})}$$

(c)
$$\begin{cases} y'' + \lambda y + \varepsilon y^2 = 0 \\ y(0) = y(\pi) = 0 \end{cases}$$

(i) $\varepsilon = 0 \Rightarrow y = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$

$y(0) = A = 0$, $y(\pi) = B\sin\sqrt{\lambda}\pi = 0$ $\boxed{3}$

$\Rightarrow \sqrt{\lambda} = n$, $n = 1, 2, \ldots$

$\Rightarrow$ eig values $\lambda_k = k^2$ $\boxed{3}$

eig functions $y_k = \sin kx$

(ii) $y \sim y_1 + \varepsilon y_{11} \cdots$, $\lambda = \lambda_1 + \varepsilon \lambda_{11} \cdots$

w/ $y_1 = \sin x$, $\lambda_1 = 1$

$$(y_1'' + \varepsilon y_{11}'' \cdots) + (\lambda_1 + \varepsilon\lambda_{11} \cdots)(y_1 + \varepsilon y_{11} \cdots) + \varepsilon(y_1 + \varepsilon y_{11} \cdots)^2 = 0$$

<u>$O(1)$</u> already solved

<u>$O(\varepsilon)$</u>
$$\underbrace{y_{11}'' + \lambda_1 y_{11}}_{\text{call } Ly_{11}} = \underbrace{-\lambda_{11} y_1 - y_1^2}_{\text{call } f(x;\lambda_{11})} \qquad (\star)$$

BC: $y_{11}(0) = y_{11}(\pi) = 0$ $\boxed{3}$

$Ly_{11} = 0$ | has a non-trivial soln,
$y_{11} = 0$ at $0, \pi$ | namely $y_1 = \sin x$ $\boxed{3}$

By FAT, for $(\star)$ to have a soln

it must hold that $\langle f, y_1 \rangle = 0$

ie $$\int_0^\pi -(\lambda_{11} y_1 + y_1^2) \cdot y_1 \, dx = 0$$

$$\Rightarrow \text{need } \lambda_{11} = \frac{-\int_0^\pi y_1^3 \, dx}{\int_0^\pi y_1^2 \, dx}$$ $\boxed{2}$

Part A: Differential Equations 2 HT2015

Trinity 2015 Exam

Problem Summary

1. a. Bookwork
   b. Similar
   c. New twist on Greens function
2. a. Bookwork
   b. Similar
   c. New
3. a. Bookwork
   b. Similar
   c. (i) Bookwork
   (ii) New