1

(a) $y''' + p_2 y'' + p_1 y' + p_0 y = 0$

i) y_1, y_2, y_3 are lin. dependent if $\boxed{1}$

$c_1 y_1(x) + c_2 y_2(x) + c_3 y_3(x) \equiv 0$ for non-trivial constants c_1, c_2, c_3

$\Rightarrow c_1 y_1' + c_2 y_2' + c_3 y_3' \equiv 0$

$\Rightarrow c_1 y_1'' + c_2 y_2'' + c_3 y_3'' \equiv 0$ $\boxed{2}$

which can be written
$$\begin{pmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This has a non-trivial soln if the determinant $= 0$,

ie if $W(y_1, y_2, y_3) = y_1(y_2' y_3'' - y_2'' y_3') - y_2(y_1' y_3'' - y_1'' y_3')$
$+ y_3(y_1' y_2'' - y_1'' y_2') \equiv 0$ $\boxed{2}$

Wronskian

Hence if $W \neq 0$, the $y_i(x)$ are lin. independent

ii) $y''' + 4y' = 0$

Let $w = y'$ $\Rightarrow$ $w'' + 4w = 0$ $\Rightarrow$ $w = \tilde{c}_1 \cos 2x + \tilde{c}_2 \sin 2x$

$\Rightarrow$ $y(x) = c_0 + c_1 \cos 2x + c_2 \sin 2x$ $\boxed{2}$

Here $y_1 = 1$, $y_2 = \cos 2x$, $y_3 = \sin 2x$

So $W(y_1, y_2, y_3) = 1 \cdot \left((-2 \sin 2x) \cdot (-4 \sin 2x) - (-4 \cos 2x) \cdot (2 \cos 2x) \right)$

$= 8$ ✓ $\boxed{2}$

(b)

$$(\star)\begin{cases} Ly \equiv y''(x) + y(x) = f(x), \quad 0 < x < a \\ y(0) = 1, \quad y(a) = 0 \end{cases}$$

(i) $(\star)$ will have a unique sol'n if there is no zero eig.-value, by FAT, i.e. if

$$\begin{cases} y'' + y = 0 \\ y(0) = 0, \quad y(a) = 0 \end{cases}$$

has only trivial sol'n. (concept) $\boxed{2}$

$y = A\cos x + B\sin x$, $y(0) = A = 0$

$y(a) = B\sin a = 0$, so non-trivial sol'n

if $a = n\pi$, $n \in \mathbb{N}$

Thus, if $a \neq \pi, 2\pi, 3\pi, \ldots$ then $(\star)$ will have a unique sol'n. $\boxed{2}$

(ii) For $a = \pi$, $y_0 = \sin x$ solves the homog. problem.

Fredholm Alt Thm: no unique sol'n $\boxed{1}$

To get solvability cond, multiply $(\star)$ by y_0 and integrate:

$$\int_0^\pi (y'' + y) y_0 \, dx = \int_0^\pi f(x) y_0 \, dx$$

$$\text{LHS} = y' y_0 - y y_0' \Big|_0^\pi + \int y \cancel{(y_0'' + y_0)}^{\,0} dx \quad \boxed{1}$$

$$= \overset{1}{\overset{\shortparallel}{y(0)}}\, y_0'(0) = \cos 0 = 1$$

Thus, solvability cond is $\boxed{\int_0^{\pi} f(x)\sin x\, dx = 1}$ $\boxed{1}$

(iii) $a=1 \Rightarrow \exists$ unique sol'n.

$My \equiv \int_0^1 g(x,\xi)\, y(\xi)\, d\xi$. We want $\{\mu, y(x)\}$ st

$My = \mu y$.

Consider $Ly = \lambda y \Rightarrow \begin{cases} y'' + (1-\lambda)y = 0 \\ y(0) = y(1) = 0 \end{cases}$

$\underline{\lambda < 1}$:
$$y = A\cos\sqrt{1-\lambda}\,x + B\sin\sqrt{1-\lambda}\,x$$

$y(0) = A = 0 \quad y(1) = 0 \Rightarrow \sqrt{1-\lambda} = n\pi \Rightarrow \lambda_n = 1 - n^2\pi^2$

are eig. vals of L, w/ $y_n = \sin n\pi x$ $\boxed{3}$

Now, in Green's fn approach, $Ly = \lambda y$ has

sol'n (where $\lambda y =$ "$f(x)$")

$$y(x) = \int_0^1 g(x,\xi)\,\lambda y\, d\xi = \lambda \underbrace{\int_0^1 g(x,\xi)\, y(\xi)\, d\xi}_{\overset{\shortparallel}{My}}$$ $\boxed{3}$

Thus, $My = \mu y$ has sol'ns $\mu_n = \dfrac{1}{\lambda_n} = \dfrac{1}{1-n^2\pi^2}$. $\boxed{3}$

w/ same eig fn's.

2

(a) $y'' - 2xy' + 2ny = 0$

(i) For Sturm-Liouville form, multiply by $\mu(x)$, st

$$\mu y'' - 2x\mu y' = \frac{d}{dx}(\mu y')$$

$\Rightarrow$ need $\mu' = -2x\mu \Rightarrow \mu = e^{-x^2}$ $\boxed{3}$

SL form is thus $\frac{d}{dx}\left(e^{-x^2} y'\right) + 2ne^{-x^2} y = 0$

(ii) Define $Ly \equiv \frac{d}{dx}\left(e^{-x^2} y'\right)$ so that y_n is an eig fn of $Ly = -2ne^{-x^2} y$ w/ eig. val. n. $\boxed{2}$

By properties of Sturm-Liouville, eig. fns y_n, y_m w/ $n \neq m$ will be orthogonal. However, since no bdy cond. given, we need to establish a domain. Supp. $a < x < b$, and consider

$$\int_a^b \underbrace{Ly_n}_{= -2ne^{-x^2} y_n} \cdot y_m \, dx = e^{-x^2} y_n'(x) y_m(x)\Big|_a^b - e^{-x^2} y_m' y_n\Big|_a^b + \int_a^b y_n \underbrace{Ly_m}_{= -2me^{-x^2} y_m} dx$$

$\boxed{4}$

Since y_n, y_m are polynomials, the bd'y terms will vanish if $a \to -\infty$, $b \to \infty$. Then we have

$$2(n-m)\int_{-\infty}^{\infty} y_n(x)\,y_m(x)\,e^{-x^2}\,dx = 0. \qquad \boxed{2}$$

Thus for $n \neq m$ we have the orthog. relation

$$\int_{\mathbb{R}} y_n(x)\,y_m(x)\,e^{-x^2}\,dx = 0$$

(b) $\sin x \, y'' + 2(x-1)y' + \frac{2}{x} y = 0$

i) Singular pts are $x = n\pi$, $n \in \mathbb{Z}$

The pt x_0 is regular if the functions

$$P(x) = \frac{2(x-1)}{\sin x} \cdot (x - x_0), \quad Q(x) = \frac{2}{\sin x \cdot x} \cdot (x - x_0)^2$$

are analytic near $x = x_0$.

Near $x = n\pi$, $\sin x = (-1)^n (x - n\pi) + O((x-n\pi)^3)$ $\boxed{3}$

Hence $P(x)$ is analytic for all $x = n\pi$, and so is $Q(x)$

$$\left[P(x) \sim 2(x-1) + O((x-x_0)^2) \right.$$

$$\left. Q(x) \sim \frac{2(x-x_0)}{x} + O((x-x_0)^3) \right]$$

as $x \to x_0$

$\therefore$ The sing. pts are all regular.

Now check $x = \infty$: let $t = \frac{1}{x}$, $w(t) = y(x)$

$$\frac{d}{dx} = -t^2 \frac{d}{dt}, \quad \frac{d^2}{dx^2} = t^4 \frac{d^2}{dt^2} + 2t^3 \frac{d}{dt}$$ $\boxed{3}$

$$\Rightarrow \sin\left(\frac{1}{t}\right)\left(t^4 w'' + 2t^3 w'\right) + 2\left(\frac{1}{t} - 1\right)\left(-t^2 w'\right) + 2t\,w = 0$$

Here $t = 0$ is an irreg. singular point, since eg

$$\frac{2t \cdot t^2}{\sin\left(\frac{1}{t}\right) t^4}$$ is <u>not</u> analytic near $t = 0$

$\therefore$ $x = \infty$ irreg sing. pt

(iii) One sol'n given by $y_1(x) = \sum_{k=0}^{\infty} a_k x^{k+\alpha_1} = \sum_{k=0}^{\infty} a_k x^{k+2}$

For 2^{nd} sol'n, since the α_i differ by an integer, there may or may not be another Frobenius sol'n.

To check, we try $y_2(x) = \sum_{k=0}^{\infty} b_k x^{k+1} = b_0 x + b_1 x^2 + \ldots$

and see if there is a contradiction at b_1, assuming $b_0 \neq 0$.

$\boxed{2}$

Plugging in: (to original ODE, eg)

$$\left(x - \frac{x^3}{6} \ldots\right)\left(b_0 x + b_1 x^2 \ldots\right)'' + (2x-2)\left(b_0 x + b_1 x^2 \ldots\right)' + \frac{2}{x}\left(b_0 x + b_1 x^2 \ldots\right) = 0$$

At x^0: $0 = 0$ ✓

At x^1: $2b_1 + 2b_0 - 4b_1 + 2b_1 = 0 \Rightarrow 2b_0 = 0 \rightarrow\leftarrow$

Contradiction $\Rightarrow$ 2^{nd} sol'n is of form (Case II b in notes):

$$y_2(x) = y_1(x) \ln x + \sum_{k=0}^{\infty} c_k x^{k+1}$$

(ii) To find the indicial eq'n for $x=0$, use

$$\lim_{x\to 0} \frac{2(x-1)}{\sin x}\cdot x = -2 \quad , \quad \lim_{x\to 0} \frac{2}{\sin x \cdot x}\cdot x^2 = 2$$

Thus, writing $P(x) = \sum_{i=0}^{\infty} P_i x^i$, $Q(x) = \sum_{i=0}^{\infty} Q_i x^i$

we have $P_0 = -2$, $Q_0 = 2$

The ODE is of form $x^2 y'' + xP(x)y' + Q(x)y = 0$

Hence $y = x^\alpha$ gives at lowest order

$\alpha(\alpha-1) + P_0\alpha + Q_0 = 0$ $\leftarrow$ Indicial Eq'n

$\Rightarrow \alpha^2 - 3\alpha + 2 = (\alpha-2)(\alpha-1) = 0$

$\Rightarrow \alpha_1 = 2$, $\alpha_2 = 1$ $\leftarrow$ Indicial Exponents

$\boxed{3}$

3 (a) $\tan(\pi+\varepsilon) = \dfrac{\sin(\pi+\varepsilon)}{\cos(\pi+\varepsilon)}$

$$\sin(\pi+\varepsilon) \sim -\varepsilon + \frac{\varepsilon^3}{6} + O(\varepsilon^5)$$

$$\cos(\pi+\varepsilon) \sim -1 + \frac{\varepsilon^2}{2} + O(\varepsilon^4) \Rightarrow \cos(\pi+\varepsilon)^{-1} \sim -1 - \frac{\varepsilon^2}{2} + O(\varepsilon^4)$$

$$\Rightarrow \tan(\pi+\varepsilon) \sim \left(-\varepsilon + \frac{\varepsilon^3}{6} + O(\varepsilon^5)\right)\left(-1 - \frac{\varepsilon^2}{2} + O(\varepsilon^4)\right) \quad \boxed{3}$$

$$= \varepsilon + \left(\frac{1}{2} - \frac{1}{6}\right)\varepsilon^3 + O(\varepsilon^5) = \varepsilon + \frac{1}{3}\varepsilon^3 + O(\varepsilon^5)$$

$\boxed{2}$

(b)
$$\begin{cases} xy' + y = \varepsilon y^2, & x > 1 \\ y(1) = 1 + \varepsilon^{\frac{1}{2}} \end{cases}$$

Seek expansion $y \sim y_0 + \varepsilon^{\frac{1}{2}} y_1 + \varepsilon y_2 + \cdots$

$\underline{O(1)}$ $\quad xy_0' + y_0 = 0 \quad$ w/ $\quad y_0(1) = 1$

$\Rightarrow \ln y_0 = -\ln x + c \rightarrow y_0 = \frac{1}{x}$ $\quad \boxed{2}$

$\underline{O(\varepsilon^{\frac{1}{2}})}$ $\quad xy_1' + y_1 = 0$

$y_1(1) = 1$

Same system $\rightarrow y_1 = \frac{1}{x}$ $\quad \boxed{2}$

$\Rightarrow y \sim \frac{1}{x} + \varepsilon^{\frac{1}{2}} \cdot \frac{1}{x} + O(\varepsilon)$

$\underline{O(\varepsilon)}$ $\quad y_2$ satisfies
$$\begin{cases} xy_2' + y_2 = y_0^2 = \frac{1}{x^2} \\ y_2(1) = 0 \end{cases} \quad \boxed{2}$$

$\underline{O(\varepsilon^{3/2})}$ $\quad y_3$ satisfies
$$\begin{cases} xy_3' + y_3 = 2y_0y_1 = \frac{2}{x^2} \\ y_3(1) = 0 \end{cases} \quad \boxed{2}$$

(c) $$\begin{cases} \varepsilon y'' + p(x) y' + y = 0 \\ y(-1) = 0, \quad y(1) = 1 \end{cases}$$

For <u>outer sol'n</u>, try $y \sim y_0 + \varepsilon y_1 \ldots$

$$\rightarrow \quad p(x) y_0' + y_0 = 0 \quad \Rightarrow \quad y_0 = C e^{-\int \frac{1}{p(x)} dx}$$

$\boxed{2}$

Need to know where b'dy layer is. Supp. layer is

at $x = x_0$ ($x_0 = \pm 1$ TBD)

Set $x = x_0 + \varepsilon^\alpha X$, define $Y(X) = y(x)$

$$\rightarrow \quad \varepsilon^{1-2\alpha} Y'' + \underbrace{\left(p(x_0) + \varepsilon^\alpha X p'(x_0) + \ldots\right)}_{\text{[expanding } p \text{ about } x_0]} \varepsilon^{-\alpha} Y' + Y = 0$$

$\boxed{4}$

To balance, we need $1 - 2\alpha = -\alpha \Rightarrow \alpha = 1$

Then expanding $Y \sim Y_0 + \varepsilon Y_1 + \ldots$ gives

So ODE reads $Y'' + p(x_0) Y' + \varepsilon\left(X p'(x_0) Y' + Y\right) + \ldots = 0$

$$\Rightarrow \quad Y_0'' + p(x_0) Y_0' = 0 \quad \Rightarrow \quad Y_0 = c_1 + c_2 e^{-p(x_0) X}$$

If <u>$x_0 = 1$</u>, then matching implies $X \to -\infty$

and $Y_0 \nearrow \infty$ since $p(1) = 1$

so matching won't work.

If $\underline{x_0 = -1}$, matching as $X \to \infty$, so

$$e^{-p(x_0) X} \to 0$$

Hence b'dy layer is at $x_0 = -1$

$\Rightarrow$ impose BC $[\, x = -1 \Rightarrow X = 0 \,]$:

$$Y_0(0) = 0 \Rightarrow Y_0 = c_1 \left(1 - e^{-X}\right)$$ $\boxed{2}$

Outer : impose $y_0(1) = 1 \Rightarrow$ ~~y_0~~ $c = e^{\int \frac{1}{p(x)} dx \big|_{x=1}}$

$$\Rightarrow y_0 = e^{-\int_1^x \frac{1}{p(s)} ds}$$

[Better route : $\frac{1}{y_0} dy_0 = -\frac{1}{p(x)} dx$, $\int_1^x dx$ both sides $\to$

$$\ln y_0 \Big|_1^x = -\int_1^x \frac{1}{p} ds$$

$$\to y_0 = e^{-\int_1^x \frac{1}{p} ds}$$]

To find c_1, require $\lim_{X \to \infty} Y_0 = \lim_{x \to -1} y_0$

$$\Rightarrow c_1 = e^{+\int_{-1}^{1} \frac{1}{p(s)} ds}$$ $\boxed{4}$

$$\text{so} \quad y_{inner}(x) = e^{\int_{-1}^{1} \frac{1}{p} ds} \left(1 - e^{-\frac{x+1}{\varepsilon}}\right)$$