DEs II Q^{n} 1

(a) $\mathcal{L}\, e^{mx} = F(m)\, e^{mx}$

So $\mathcal{L}\, e^{\alpha x} = F(\alpha)\, e^{\alpha x} = 0$

and $\frac{\partial}{\partial m}\left[\mathcal{L}\, e^{mx}\right] = \mathcal{L}\left[\frac{\partial}{\partial m} e^{mx}\right] = \mathcal{L}\left[x e^{mx}\right]$

$\therefore\ \mathcal{L}\left[x e^{mx}\right] = \left(F'(m) + x F(m)\right) e^{mx}$

$\therefore\ \mathcal{L}\left[x e^{\alpha x}\right] = \left(F'(\alpha) + x F(\alpha)\right) e^{\alpha x} = 0$

[4]

unseen - analogous to “differentiation method”

1(b) $$\boxed{\begin{array}{c} \mathcal{L}y = y'' + 6y' + 9y = f \\ y(0) = 0, \quad y(1) = 0 \end{array}}$$

Green's function satisfies

$$\boxed{\begin{array}{c} g_{xx} + 6g_x + 9g = \delta(x-\xi) \\ g = 0 \quad \text{at} \quad x = 0, 1 \end{array}}$$ [2]

So we have $g_{xx} + 6g_x + 9g = 0$ for $x \neq \xi$

Auxiliary eq^{n} $(m+3)^2 = 0$

So general solution $g = c_1 e^{-3x} + c_2 x e^{-3x}$

Given BCs: $g(x,\xi) = \begin{cases} A(\xi)\, x\, e^{-3x} & 0 < x < \xi < 1 \\ B(\xi)(1-x)\, e^{-3x} & 0 < \xi < x < 1 \end{cases}$ [2]

At $x = \xi$, jump conditions $\begin{cases} [g]_{x=\xi_-}^{\xi_+} = 0 \\ [g_x]_{x=\xi_-}^{\xi_+} = 1 \end{cases}$

give $B(1-\xi)e^{-3\xi} = A\xi e^{-3\xi}$

$B(3\xi-4)e^{-3\xi} - A(1-3\xi)e^{-3\xi} = 1$ [2]

ie. $\begin{pmatrix} \xi & -(1-\xi) \\ (1-3\xi) & (4-3\xi) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ -e^{3\xi} \end{pmatrix}$

Determinant is $\xi(4-3\xi) + (1-\xi)(1-3\xi) = 1$

So
$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 4-3\xi & (1-\xi) \\ -(1-3\xi) & \xi \end{pmatrix} \begin{pmatrix} 0 \\ -e^{3\xi} \end{pmatrix} = \begin{pmatrix} -(1-\xi)e^{3\xi} \\ -\xi e^{3\xi} \end{pmatrix}$$

$$\therefore \boxed{g(x,\xi) = \begin{cases} -x(1-\xi)e^{3(\xi-x)} & 0<x<\xi<1 \\ -(1-x)\xi e^{3(\xi-x)} & 0<\xi<x<1 \end{cases}}$$

[2]

Solution to inhomogeneous BVP is then

$$\boxed{y(x) = \int_0^1 g(x,\xi) f(\xi) d\xi}$$

similar to problem sheet

1(b) $\boxed{\begin{array}{l} \mathcal{L}y + \lambda y = y'' + 6y' + (9+\lambda)y = 0 \\ \text{with} \quad y(0) = y(1) = 0 \end{array}}$

Auxiliary eq^n: $(m+3)^2 + \lambda = 0$

$\Rightarrow \quad m = -3 \pm i\sqrt{-\lambda}$ (assuming $\lambda < 0$)

Given BCs: $\boxed{\begin{array}{l} y_n(x) = e^{-3x} \sin(n\pi x) \\ \text{with} \quad \lambda_n = -n^2\pi^2 \end{array}}$ $n = 1, 2, 3, \ldots$ [2]

Adjoint operator $\boxed{\mathcal{L}^* w = w'' - 6w' + 9w}$

then

$$w\mathcal{L}y - y\mathcal{L}^*w = w(y'' + 6y' + 9y) - y(w'' - 6w' + 9w)$$

$$= (wy' - yw' + 6wy)'$$

So $\int_0^1 (w\mathcal{L}y - y\mathcal{L}^*w)\,dx = [wy' - yw' + 6wy]_0^1$

[Given BCs] $= w(1)y'(1) - w(0)y'(0)$

To make this $\equiv 0$, impose adjoint BCs $\boxed{w(0) = w(1) = 0}$

then $\boxed{\begin{array}{l} \mathcal{L}^* w + \lambda w = w'' - 6w' + (9+\lambda)w = 0 \\ \text{with} \quad w(0) = w(1) = 0 \end{array}}$ [2]

has solutions (as above) $\boxed{w_n(x) = e^{3x} \sin(n\pi x)}$

$n = 1, 2, \ldots$ [2]

We have $\mathcal{L}y = f$

$$\therefore \quad \int_0^1 \mathcal{L}y(x)\, w_n(x)\, dx = \langle \mathcal{L}y, w_n\rangle = \langle f, w_n\rangle$$

$$\therefore \quad \langle f, w_n\rangle = \langle y, \mathcal{L}^* w_n\rangle = +\lambda_n \langle y, w_n\rangle$$

[NB all $\lambda_n \neq 0$]

Write $y(x) = \sum_{i=1}^{\infty} c_i\, y_i(x)$

then $\langle f, w_n\rangle = +\lambda_n \sum_{i=1}^{\infty} c_i \langle y_i, w_n\rangle$

Note $\langle y_i, w_n\rangle = \int_0^1 \sin(i\pi x)\sin(n\pi x)\, dx = \frac{1}{2}\delta_{in}$

so $\langle f, w_n\rangle = +\frac{\lambda_n c_n}{2}$

$$\therefore \quad c_n = +\frac{2}{\lambda_n}\langle f, w_n\rangle$$ [2]

& $$\boxed{y(x) = -\frac{2}{\pi^2}\sum_{n=1}^{\infty} \frac{e^{-3x}\sin(n\pi x)}{n^2}\int_0^1 f(\xi)\, e^{3\xi}\sin(n\pi\xi)\, d\xi}$$

similar to problem sheet

1(d) from (c), we know that $\lambda_1 = -\pi^2$ is an eigenvalue of $\mathcal{L}$, with eigenvectors

$$y_1(x) = e^{-3x} \sin(\pi x), \quad w_1(x) = e^{3x} \sin(\pi x)$$

[1]

Note

$$\langle \mathcal{L} y + \pi^2 y, w \rangle - \langle y, \mathcal{L}^* w + \pi^2 w \rangle$$

$$= \int_0^1 (y'' + 6y' + 9y) w - (w'' - 6w' + 9w) y \, dx$$

$$= \left[y'w - w'y + 6yw \right]_0^1$$

[2]

given $\mathcal{L}^* w_1 + \pi^2 w_1 = 0$,

$$\langle f, w_1 \rangle = \left[y' w_1 - w_1' y + 6 y w_1 \right]_0^1$$

with $w_1(0) = w_1(1) = 0$

$$w_1'(x) = \left[3 \sin(\pi x) + \pi \cos(\pi x) \right] e^{3x}$$

$$w_1'(0) = \pi, \quad w_1'(1) = -\pi e^3$$

so we have the solvability condition:

$$\boxed{\int_0^1 f(x) \sin(\pi x) e^{3x} \, dx = \pi + 2\pi e^3}$$

[2]

new example - inhomogeneous BCs make it tricky…

2(a) $\boxed{(1-x^2)y'' - xy' + \lambda^2 y = 0}$

let $x = \pm 1 + X$

$$(\mp 2X - X^2)\,y_{XX} + (\mp 1 - X)\,y_X + \lambda^2 y = 0$$

i.e. $(2X \pm X^2)\,y_{XX} + (1 \pm X)\,y_X \mp \lambda^2 y = 0$

write as $y'' + P_1 y' + P_0 y = 0$

with $P_1 = \dfrac{1 \pm X}{2X \pm X^2}$, $P_0 = \dfrac{\mp \lambda^2}{2X \pm X^2}$

so P_1 & P_0 are not analytic at $X=0$, but XP_1 and $X^2 P_0$ are.

$\therefore$ $x = \pm 1$ are regular singular points. [2]

If $y \sim X^k$ as $X \to 0$, then...

$$2k(k-1)X^{k-1} + kX^{k-1} + \dots \sim 0$$

i.e. $\boxed{k(2k-1) = 0}$ is the indicial eq^{n}.

[2]

& local behavior is

$$y \sim C_1(1 + a_1 X + \dots) + C_2\sqrt{X}\,(1 + b_1 X + \dots)$$

as $X = x \mp 1 \to 0$ [2] standard example

To analyse $x = \infty$, let $\underline{x = 1/X}$

$$d/dx = -X^2 \, d/dX$$

$$\left(1 - \frac{1}{X^2}\right) X^2 \frac{d}{dX}\left[X^2 \frac{dy}{dX}\right] + \frac{1}{X} \cdot X^2 \frac{dy}{dX} + \lambda^2 y = 0$$

$$(X^2 - 1)\left[X^2 \frac{d^2y}{dX^2} + 2X \frac{dy}{dX}\right] + X \frac{dy}{dX} + \lambda^2 y = 0$$

$$(1 - X^2)\frac{d^2y}{dX^2} + \frac{(1 - 2X^2)}{X}\frac{dy}{dX} - \frac{\lambda^2}{X^2} y = 0$$

[2]

again, it's a regular singular point.

with $y \sim X^k$ as $X \to 0$..

$$k(k-1) X^{k-2} + k X^{k-2} - \lambda^2 X^{k-2} \sim 0$$

$\boxed{k^2 - \lambda^2 = 0}$ is the indicial equation

[2]

So general behavior is

$$y \sim c_1 x^{\lambda}\left(1 + \frac{a_1}{x} + \ldots\right) + c_2 x^{-\lambda}\left(1 + \frac{b_1}{x} + \ldots\right)$$

as $x \to \infty$ standard example [1]

If 2λ is an integer, then the indices differ by an integer. In this case, the series with $k = +\lambda$ is unchanged but the series with $k = -\lambda$ might encounter a contradiction in the coefficient of X^{λ}. If so, then we need to introduce a logarithmic term of the form $Y = \log X \, Y_1(X) + Y_2(X)$. bookwork

[2]

2(b) $(1-x^2)y'' - xy' + \lambda^2 y = 0$

$x=0$ is an ordinary point so try regular expansion

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$\Rightarrow \sum_{n=0}^{\infty} \left[n(n-1)a_n\left(x^{n-2} - x^n\right) - n a_n x^n + \lambda^2 a_n x^n \right] = 0$$

The first term vanishes when $n=0$ or 1, so we can write

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} - (n^2-\lambda^2)a_n \right] x^n = 0$$

For this to be true $\forall n$, we must have

$$\boxed{a_{n+2} = \frac{(n^2-\lambda^2)}{(n+2)(n+1)} a_n} \quad \text{for } n=0,1,\ldots$$

[4]

Given a_0 & a_1, this relation determines $a_2, a_4, \ldots$ and $a_3, a_5, \ldots$

There is a polynomial solution iff the series terminates, i.e. either $a_{2k} = 0 \ \forall k > m$ or $a_{2k+1} = 0 \ \forall k > m$ } for some m.

This occurs iff $\lambda = n$ for some $n = 0, 1, \ldots$

(given $\lambda \geq 0$) standard example [2]

2(c) $(1-x^2)y'' - xy' + \lambda^2 y$

$$\Rightarrow \quad \boxed{\left[\sqrt{1-x^2}\, y'\right]' + \frac{\lambda^2 y}{\sqrt{1-x^2}} = 0}$$ [2]

Consider two polynomial solutions with $m \neq n$:

$$\left[\sqrt{1-x^2}\, P_m'\right]' + \frac{m^2 P_m}{\sqrt{1-x^2}} = 0 \quad (1)$$

$$\left[\sqrt{1-x^2}\, P_n'\right]' + \frac{n^2 P_n}{\sqrt{1-x^2}} = 0 \quad (2)$$

$P_n(1) - P_m(2)$:

$$(m^2-n^2)\frac{P_m P_n}{\sqrt{1-x^2}} = P_m\left[\sqrt{1-x^2}\, P_n'\right]' - P_n\left[\sqrt{1-x^2}\, P_m'\right]'$$

$$= \left(\sqrt{1-x^2}\left(P_m P_n' - P_n P_m'\right)\right)'$$

$$\therefore \quad (m^2-n^2)\int_a^b \frac{P_m(x) P_n(x)}{\sqrt{1-x^2}}\,dx = \left[\sqrt{1-x^2}\left(P_m P_n' - P_n P_m'\right)\right]_a^b$$ [2]

Choose $\boxed{a=-1,\ b=+1}$. Then:
RHS $= 0$ because P_m, P_n are polynomials & hence bounded as $x \to \pm 1$. So for $m \neq n$,

$$\boxed{\int_{-1}^{1} \frac{P_m(x) P_n(x)\,dx}{\sqrt{1-x^2}} = 0}$$ [2]

unconventional example including finding limits

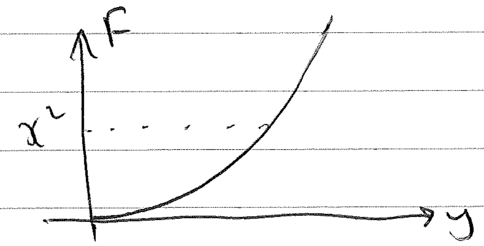
3(a) $$\boxed{\frac{\varepsilon y^2}{1+xy^2} + x^2 y = x^2}$$ with $x>0$
$0<\varepsilon\ll 1$

$$\text{LHS} = F(y; x, \varepsilon) \quad \text{(say)}$$
$$= \frac{\varepsilon}{x}\left[1 - \frac{1}{1+xy^2}\right] + x^2 y$$

so $$\frac{\partial F}{\partial y} = \frac{2\varepsilon y}{(1+xy^2)^2} + x^2 > 0 \quad \text{for } y>0$$

and $F\to 0$ as $y\to 0$, $F\to\infty$ as $y\to\infty$.

So



for any $x^2>0$, there is a unique positive root for y. [2]

If $y \sim y_0(x) + \varepsilon y_1(x) + \ldots$

then $\boxed{y_0 = 1}$ & $$x^2 y_1(x) + \frac{y_0^2(x)}{1+x y_0^2(x)} = 0$$

$$\therefore \boxed{y_1(x) = -\frac{1}{x^2(1+x)}}$$

So $$\boxed{y(x;\varepsilon) \sim 1 - \frac{\varepsilon}{x^2(1+x)}}$$ [2]

The expansion fails if the second term stops being much smaller than the first term, which happens when $\boxed{x = O(\sqrt{\varepsilon})}$ [1]

To examine this distinct region, rescale $x = \sqrt{\varepsilon} X$ to get

$$\boxed{\frac{Y^2}{1+\sqrt{\varepsilon} X Y^2} + X^2 Y = X^2}$$ where $y(x;\varepsilon) = Y(X;\varepsilon)$

To leading order $Y_0^2 + X^2 Y_0 - X^2 = 0$

$$\therefore \quad Y_0 = \frac{1}{2}\left[-X^2 \pm \sqrt{X^4 + 4X^2}\right]$$

positive solution: $\boxed{Y_0 = \frac{1}{2}\left[-X^2 + X\sqrt{X^2+4}\right]}$

[2]

As $X \to \infty$: $Y_0 = \frac{1}{2}X^2\left[-1 + \sqrt{1 + \frac{4}{X^2}}\right]$

$$\sim \frac{1}{2}X^2\left[-1 + \left(1 + \frac{2}{X^2} + \dots\right)\right]$$

i.e. $Y_0 \to 1$ as $X \to \infty$

which does match with leading-order outer $y = 1$ ☒

[2]

new example - trickyish matching

3(b)

$$\boxed{\begin{gathered} y'' + \left(1 + \varepsilon\lambda + \varepsilon(y')^2 - \varepsilon y^2\right) y = 0 \\ y(0) = 0, \qquad y(\pi) = 0 \end{gathered}}$$

write $y \sim y_0 + \varepsilon y_1 + \dots$

then

$$\boxed{\begin{gathered} y_0'' + y_0 = 0, \\ y_0(0) = 0, \quad y_0(\pi) = 0 \end{gathered}}$$

&

$$\boxed{\begin{gathered} y_1'' + y_1 = \left[y_0^2 - (y_0')^2 - \lambda\right] y_0 \\ y_1(0) = 0, \quad y_1(\pi) = 0 \end{gathered}}$$

The leading-order problem is satisfied by

$\boxed{y_0(x) = A\sin(x)}$ where A is apparently arbitrary. [2]

then
$$y_1'' + y_1 = \left[A^2\sin^2 x - A^2\cos^2 x - \lambda\right] A\sin x$$
$$= 2A^3\sin^3 x - (\lambda + A^2) A\sin x$$
$$= \frac{A^3}{2}\left[3\sin x - \sin(3x)\right] - (\lambda A + A^3)\sin x$$
[using hint]
$$= \frac{1}{2}A^3\sin x - \lambda A\sin x - \frac{A^3}{2}\sin(3x) = R(x), \text{ say.}$$
[3]

Since $\mathcal{L}y_0 = 0$ has nontrivial solution,

$\mathcal{L}y_1 = R$ is solvable only if R is orthogonal to kernel of $\mathcal{L}^*$.

Here L (and BCs) is self-adjoint, so solvability condition is $\langle R, \sin x \rangle = 0$

i.e. $\boxed{A(A^2 - 2\lambda) = 0}$ [3]

new example - unfamiliar use of solvability

3(c)

$$\boxed{\begin{gathered}\varepsilon y'' - (1+x^2)y' + xy = 0 \\ y(0)=1, \quad y(1)=0\end{gathered}}$$

Leading-order outer $\quad (1+x^2)y_0' = xy_0$

$$\Rightarrow \frac{y_0'}{y_0} = \frac{x}{1+x^2}$$

$$\Rightarrow \boxed{y_0(x) = A\sqrt{1+x^2}} \quad \text{for some constant } A.$$

$y(0)=1$ gives $A=1$, $y(1)=0$ gives $A=0$.
So, it's impossible to satisfy both.

Since $(1+x^2)>0$, the boundary layer is at the right-hand boundary $x=1$.

So we impose the BC $y(0)=1$ on the outer to get

$$\boxed{y_0(x) = \sqrt{1+x^2}} \qquad [3]$$

B-layer analysis $\quad x = 1-\varepsilon X, \quad y(x) = Y(X)$

gives
$$Y'' + (2-2\varepsilon X+\varepsilon^2X^2)Y' + \varepsilon(1-\varepsilon X)Y = 0$$

At leading order $\quad Y_0'' + 2Y_0' = 0$

BC $\quad Y_0(0) = 0$

matching $\quad Y_0(X) \to \sqrt{2}$ as $X \to \infty$

Solution $\boxed{Y_0(x) = \sqrt{2}\left(1 - e^{-2x}\right)}$ [3]

Then $y'(1) = -\frac{1}{\varepsilon} Y_0'(0) \sim -\frac{1}{\varepsilon} Y_0'(0)$

$\therefore$ $\boxed{y'(1) \sim -\frac{2^{3/2}}{\varepsilon}}$ [2]

standard example with new but easy rider