Q1

3B

(a) $$\int_a^b f(x)\,dx \approx \frac{h}{2}\left[f(a) + f(b)\right]$$

with $h = b - a$

(deriving: could construct the linear polynomial interpolant and integrate that proxy exactly)

4B

(b)(i) interpolant $p_1(x)$ has error

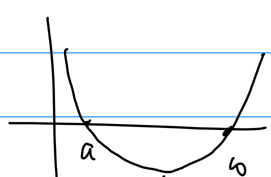
$$e(x) := f(x) - p_1(x) = \frac{\pi(x)}{(n+1)!} f''(\xi) \quad \text{for some } \xi(x)$$

where $\pi(x) = (x-a)(x-b)$.

Integrating both sides:

$$\left|\int_a^b e(x)\,dx\right| \leq \frac{1}{2!} \int_a^b \left|(x-a)(x-b)\right| \max_{\xi \in [a,b]} \left|f''(\xi)\right| dx$$

$$= \frac{1}{2} \max_{\dots} f''(\xi) \int_a^b \left|(x-a)(x-b)\right| dx$$



note $\left|(x-a)(x-b)\right| = -(x-a)(x-b)$ on $x \in [a,b]$

$$= \frac{1}{2} \max_{\dots} f''(\xi)\,(-1) \int_a^b (x-a)(x-b)\,dx$$

Lemma: $\int_a^b (x-a)(x-b) = -\frac{(b-a)^3}{6}$

Proof: IbP: $LHS = (x-b)\frac{(x-a)^2}{2}\Big|_a^b - \int_a^b \frac{(x-a)^2}{2}\,dx$

$= -\frac{(x-a)^3}{6}\Big|_a^b = -\frac{(b-a)^3}{6}$

$\boxtimes$

Sub into previous

$$|e(x)| \leq \frac{(b-a)^3}{12} \max_{\xi \in [a,b]} f''(\xi)$$

Conditions on f: f'' continuous on $[a,b]$

b) i) (cont) Cannot be improved because it is attained: as previous but w/o absolute values:

stated in notes as a "tighter bound"

$$\int_a^b e(x) = \underbrace{-\int_a^b -}\frac{(x-a)(x-b)}{2} f''(\xi(x))\,dx$$

add minus signs so can use IMVT

4B/S

different phrasing so S also was optional on sheets

$$= -\frac{1}{2} f''(\eta)\int_a^b -(x-a)(x-b)\,dx$$

for some $\eta \in (a,b)$ (when note integrand is positive)

$$= -\frac{1}{6} f''(\eta)\,(b-a)^3$$

b)ii) Exact for poly of degree 1.

3S

not a contradiction: $f'' \equiv 0$

(if someone uses odd fcn $\sin(x)$ then has inflection pt so $f''(\eta)=0$)

c)i) $$\int_{a=x_0}^{b=x_n} f(x)\,dx \approx h\left[f(x_0)+2f(x_1)+2f(x_2)+\dots+2f(x_{n-1})+f(x_n)\right]$$

when $h = x_1 - x_0$

3B

and x_i are equispaced

c)ii) Either (A) $$f(x) = \begin{cases} 2x & x < x_1 \\ 3x & x \geq x_1 \end{cases}$$

or any pw. linear polys

3N

(B) use symmetry, e.g.,

$$\int_{-h}^{h} \sin(x) = 0$$

g) Richardson Extrapolation

$$Y = \text{estimateY}(h) + ch^2 + O(h^3)$$

and $9\left[Y = \text{estimateY}(h/3) + \frac{1}{9}ch^2 + O(h^3) \right]$

5N

So $9Y - Y \doteq 9\,\text{estimateY}(h/3) - \text{estimateY}(h) + O(h^3)$

$$\Rightarrow \quad Y \approx \frac{9\,\text{estimateY}(1/3) - \text{estimateY}(1)}{8}$$

$$= \frac{9 \cdot 3 - 2}{8} = \frac{25}{8} = 3.125$$

$\boxed{2}$ a) $p_n(x_i) = f(x_i) \quad \forall i$

5B

Existence $L_{n,k}(x) = \dfrac{(x-x_0)\dots(x-x_{k-1})(x-x_{k+1})\dots(x-x_n)}{(x_k-x_0)\dots(x_k-x_{k-1})(x_k-x_{k+1})\dots(x_k-x_n)}$

by construction $L_{n,k} \in \Pi_n$

and $L_{n,k}(x_i) = \delta_{ik}$

Now $p_n(x) := \sum_{k=0}^{n} f_k L_{n,k}(x)$

and by above, this interpolates

Uniqueness: assume p, q are interpolants

Consider $d = p - q$ then d has zeros at each x_i, i.e. $n+1$ roots. But $d \in \Pi_n$. Too many roots, contradicts F.T. of Algebra.

GB

b) Trivial for $x = x_i \quad i = 0, \dots, n$

So assume $x \neq x_i$, define

$$Q(t) := e(t) - \frac{e(x)}{\pi(x)} \pi(t)$$

This has all x_i as roots and x

$\hookrightarrow$ $n+2$ roots

$Q'(t)$ vanishes at $n+1$ pts

$Q''(t)$ " " n pts

$Q^{(n+1)}(t)$ " " 1 pt

$$\boxed{Q^{(n+1)}(\xi) = 0}$$

$$Q^{n+1}(t) = \underbrace{e^{(n+1)}(t)}_{f^{(n+1)}(t) - \underbrace{0}_{\text{poly } \Pi_n}} - \frac{e(x)}{\pi(x)} \underbrace{\pi^{(n+1)}(t)}_{\text{monic, degree } n+1:\ x^{n+1} + L.O.T.\ \text{So } \pi^{(n+1)} = (n+1)!}$$

$$\text{So} \quad Q^{n+1}(\xi) = 0 = f^{(n+1)}(\xi) - \frac{e(x)}{\pi(x)} (n+1)!$$

$$\Rightarrow \boxed{e(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \pi(x)}$$

3N

c) find interpolant p_n then differentiate $\frac{d}{dx} p_n =: q(x)$

[dropped]

Note: out of order

c) ii) $$f'(x) - \underbrace{p_n'(x)}_{q(x)} = \frac{d}{dx} \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \prod (x - x_i)$$

4N

$$+ \{\text{product rule}\}$$

But chain rule on $f^{(n+1)}(\xi(x))$ needs ξ differentiable (unknown) and even if $\xi'(x)$ exists, we do not have a bound.

5S

(quite similar to (b) proof)

c) i) Let $f^{(n+1)}$ exist, bdd on $[a,b]$. Let $x_0, \ldots, x_n$ be distinct, each in $[a,b]$. note only n, not $n+1$

Then $\exists$ $\eta_i \in (a,b)$ distinct, $i = 1, \ldots, n$ and $\exists$ $q \in \Pi_{n-1}$ such that

$$f'(x) - q(x) = \frac{f^{(n+1)}(\xi)}{n!} \pi_n^*(x)$$

where $\pi_n^*(x) := (x - \eta_1) \ldots (x - \eta_n)$

and where $q = \frac{d}{dx} p_n(x)$, $p_n \in \Pi_n$ the interpolating poly.

Proof: let $p_n \in \Pi_n$ be the interpolant as above so $p_n(x) - f(x) = 0$ at x_i so $\exists$ some intermediate points $\eta_i \in (x_{i-1}, x_i)$ where deriv $e'(x_i) = 0$, ie, $f'(\eta_i) - p_n'(\eta_i) = 0$

Define $e(t) = f'(x) - p_n'(x)$ $\left(= f'(x) - q(x)\right)$

Somewhat similar to notes from here

Define $\mathcal{Q}(t) = e(t) - \dfrac{e(x)}{\hat{\pi}(x)}\hat{\pi}(t)$

$\mathcal{Q}$ has roots at each x_i and at new pt x

Rolle's Thm

$\Rightarrow$ $n+1$ pts

$\Rightarrow$ cts implies $\mathcal{Q}'$ vanishes at n points.

$\Rightarrow$ $\mathcal{Q}''$ " " $n-1$ " .

$\mathcal{Q}^{(n)}$ " " 1 point.

ie. $\mathcal{Q}^{(n)}(\xi) = 0$

Now $\mathcal{Q}^{(n)}(t) = e^{(n)}(t) - \dfrac{e(x)}{\hat{\pi}(x)}\underbrace{\hat{\pi}^{(n)}(t)}$

monic poly degree n: $x^n + L.O.T.$ $\Rightarrow n!$

$e^{(n)}(t) = f^{(n+1)}(t) - \cancel{p_n^{(n+1)}(t)} \to 0$

Thus $\mathcal{Q}^{(n)}(\xi) = 0 = f^{(n+1)}(\xi) - \dfrac{e(x)}{\hat{\pi}(x)}\cdot n!$

$\Rightarrow e(x) = \dfrac{f^{(n+1)}(\xi)}{n!}\hat{\pi}(x)$

or $f'(x) - q(x) =$ " .

c) iii) Interpolant (unique so doesn't matter how we get it) $p_1(x) := \frac{f_1 - 0}{h} x = \left(\sin(ah) + a\varepsilon\right)\frac{x}{h}$

Now $q(x) = \frac{\sin ah + a\varepsilon}{h}$

Convergence of $q(x) = a\cancel{\frac{\sin ah}{ah}} + \frac{a\varepsilon}{h}$ 1 in lim

5N

$$\lim_{h\to 0} q(x) = a + \frac{a\varepsilon}{h}$$

which is unbounded as $h \to 0$ so $q(x)$ does not converge, no matter how small ε is

that is small change to data can be a large change in the derivative

Convergence of $p_1(x)$: consider $x \in [0,h]$ then

$$p_1(x) \le \left(a\frac{\sin ah}{ah} + a\varepsilon\right)\frac{h}{h} \quad \text{largest } x$$

So $\lim_{h\to 0} p_1(x) = a + a\varepsilon$ (uniformly in x)

and we see a small ε perturbation to $f(x_1)$ makes a small change in the interpolant.

Why $\sin(ax)$? I case someone wants to discuss that the deriv. can be large even for bdd fcn, (even w/o errors)

G (but cannot get full value for this)

$\boxed{3}$ a) Orthogonal matrix: $Q^TQ = QQ^T = I$

def'n: $Q^{-1} = Q^T$

2B

$$\|Qx\|^2 = (Qx)^T Qx = (x^T Q^T) Q x$$
$$= x^T Q^T Q x = x^T x$$

3S

(or perhaps fix W as we don't concentrate on flops too much. But pretty easy and we did A=LU in class/notes)

b) $QRx = b$

$\Rightarrow Qy = b \Rightarrow y = Q^{-1}b = Q^Tb$ so this is matrix multiply $(O(n^2))$

Now solve $Rx = y$ with backsub using $O(n^2)$. Total: $O(n^2)$

c) i)

4B

$$J(i,j,\theta) = \begin{array}{c} \\ \left[\begin{array}{ccccccccc} 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & c & & s & & & \\ & & & & 1 & & & & \\ & & & & & \ddots & & & \\ & & & & & & 1 & & \\ & & & -s & & c & & & \\ & & & & & & & 1 & \\ & & & & & & & & \ddots \\ & & & & & & & & & 1 \end{array}\right] \\ \begin{array}{ccc} & \text{column } i & \text{col } j \end{array} \end{array} \begin{array}{l} \\ \\ \\ \\ \leftarrow \text{row } i \\ \\ \\ \\ \leftarrow \text{row } j \\ \\ \\ \\ \end{array}$$

where $c = \cos\theta$, $s = \sin\theta$

Orthogonal: $\langle \text{col } i, \text{col } j \rangle = -cs + cs = 0$

$k \neq i,j$ $\quad \langle \text{col } i, \text{col } k \rangle = 1\cdot 0 + 0\cdot c + 0\cdot s = 0$

$l \neq i,j,k$ $\quad \langle \text{col } j, \text{col } k \rangle = 1\cdot 0 + 0\cdot s + 0\cdot c = 0$

$\langle \text{col } k, \text{col } l \rangle = 1\cdot 0 + 1\cdot 0 = 0$

c) ii) $J(i,j,\theta)\vec{x} =: \vec{y}$

$$y_j = -s x_i + c x_j = 0 \Rightarrow \boxed{s/c = x_j/x_i}$$

$$\hookrightarrow \theta = \arctan\left(x_j/x_i\right)$$

5B

or v. easy 5

or otherwise

also, e.g.,

$$\underbrace{x_i^2 \sin^2\theta + \overset{0}{x_i^2 \cos^2\theta - x_i^2 \cos^2\theta}}_{x_i^2 = (x_i^2 + x_j^2)\cos^2\theta} = x_j^2 \cos^2\theta$$

$$\boxed{c = \frac{x_i}{\sqrt{x_i^2 + x_j^2}}} \quad \text{similarly} \quad \boxed{s = \frac{x_j}{\sqrt{x_i^2 + x_j^2}}}$$

$$y_i = c x_i + s x_j = \frac{x_i^2 + x_j^2}{\sqrt{x_i^2 + x_j^2}} = \boxed{\sqrt{x_i^2 + x_j^2} = y_i}$$

First: brute force approach:

d) i) After applying $J(1,2)$, we have $\begin{bmatrix} \sqrt{x_1^2 + x_2^2} \\ 0 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$

45

Next apply $J(1,3)$, we have $\begin{bmatrix} \sqrt{\left(\sqrt{x_1^2 + x_2^2}\right)^2 + x_3^2} \\ 0 \\ 0 \\ x_4 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \sqrt{x_1^2 + x_2^2 + x_3^2} \\ 0 \\ 0 \\ x_4 \\ \vdots \\ x_n \end{bmatrix}$

Continuing in this way, we have $Q_1 \vec{a} = \begin{bmatrix} \sqrt{x_1^2 + \dots + x_n^2} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} \|\vec{a}\|_2 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

Alt: use $\|Qx\| = \|x\|$. Much nicer, but they should show/acknowledge that Q_1 is orthog. — e.g. by showing product of two orthog. matrices is orthog.

3S/W e) ~~all entries of b are changed in general~~ (dropped)

2S d) ii) $Q_2 = J(2,n)J(2,n-1)\dots J(2,4)J(2,3)$

(operating on $Q_1[\vec{a} \mid \vec{b}]$ to define the θ's)

d) iii) Let $A = [\vec{q} \mid \vec{p}]$ then $\vec{q} \cdot \vec{p} = 0$

and $\|\vec{q}\|_2 = \|\vec{p}\|_2 = 1$

(i.e., $\vec{q}$ and $\vec{p}$ are ortho<u>normal</u>)

5W We can say $C := Q_1 A = \begin{bmatrix} 1 & 0 \\ 0 & x \\ 0 & x \\ 0 & x \end{bmatrix}$ ← question asks about these two

$\boxed{C_{11} = 1}$ proof: by d) this $C_{11} = \|\vec{q}\|_2 = 1$ ☒

$\boxed{C_{12} = 0}$ note by e) this does not happen in general: must be by orthogonality

We have $Q_1\vec{q}$ and $Q_1\vec{p}$. Now consider $(Q_1\vec{q})\cdot(Q_1\vec{p}) = (Q_1\vec{q})^T(Q_1\vec{p}) = \vec{q}^T Q_1^T Q_1 \vec{p}$

$= \vec{q}^T\vec{p} = 0$

So $Q_1\vec{p}$ is orthogonal to $Q_1\vec{q}$ but $Q_1\vec{q}$ is $[1,0,0\dots,0]^T$ so 1^{st} entry of $Q_1\vec{p}$ is zero

$\Rightarrow C_{12} = 0$ ☒