Part-A Numerical Analysis 2016 Q1 solution

Existence: Define $L_{n,k}(x) = \prod_{\substack{j=0 \\ j\neq k}}^{n} \left[(x-x_j)/(x_k-x_j)\right] \in \Pi_n$ for $k=0,1,\dots,n$

then $L_{n,k}(x_j) = \begin{cases} 1 & \text{if } j=k \\ 0 & \text{o.w.} \end{cases}$ so $p(x) = \sum_{k=0}^{n} f_k L_{n,k}(x) \in \Pi_n$

satisfies $p(x_i) = f_i \quad , i=0,1,\dots,n$ (*)

Uniqueness: Assume $p, q \in \Pi_n$ both satisfy the conditions (*), then let $r = p-q \in \Pi_n$ so that $r(x_i) = f_i - f_i = 0 \quad , i=0,1,\dots,n$.
But $r \in \Pi_n$ therefore has at least $n+1$ point where it is zero $\Rightarrow r(x) = 0 \ \forall x$.
hence uniqueness

(Bookwork) [6 marks]

If $n=2$, $x_0=0$, $x_1=1$, $x_2=2$, $f_0=0$, $f_1=-1$, $f_2=-2$ then $p(x) = -x$ clearly satisfies the condition (*) and hence by the above is the unique Lagrange Interpolating polynomial for the given data

(easy example) [2 marks]

$q(x) = \alpha x(x-1)(x-2) - x$ must be the set of all cubics which interpolate the given data for any $\alpha \in \mathbb{R}$.

$q(x) = \alpha x^3 - 3\alpha x^2 + (2\alpha-1)x$, so $q'(x) = 3\alpha x^2 - 6\alpha x + (2\alpha-1)$

So $q'(x_0) = q'(0) = 1$ only if $\alpha = 1$, so $q(x) = x^3 - 3x^2 + x$.

(unseen simple generalisation) [3 marks]

Assume $p, q \in \Pi_{n+1}$ both satisfy (*) and $p'(x_0) = q'(x_0) = f_0' \in \mathbb{R}$.

$r = p-q \in \Pi_{n+1}$ satisfies $r(x_i) = 0$, $i=0,1,\dots,n$, $r'(x_0) = 0$

Applying the Mean Value Theorem (or Rolle's Theorem) in each interval (x_{i-1}, x_i) $i=1,\dots,n$ $\exists\, \xi_i$ s.t. $r'(\xi_i) = 0 \quad i=1,\dots,n$

Hence $r' \in \Pi_n$ vanishes at ξ_i, $(i=1,\dots,n)$ and x_0 which are $n+1$ distinct points; hence $r'(x) = 0 \ \forall x$ and so r is constant, but zero at e.g. x_0 so $r \equiv 0$ and we have uniqueness [4 marks] of A Num. Anal. 2016 Q1 soln. cont.

is the quadrature rule we need functions $L_0, L_1, L_2 \in \Pi_2$ s.t.

$L_0(0)=1$, $L_0'(0)=0$, $L_0(1)=0$ $\Rightarrow$ $L_0(x) = -(x-1)(x+1)$

$L_1(0)=0$, $L_1'(0)=0$, $L_1(1)=1$ $\Rightarrow$ $L_1(x) = x^2$

$L_2(0)=0$, $L_2'(0)=1$, $L_2(1)=0$ $\Rightarrow$ $L_2(x) = -x(x-1)$

Then given $f(x_0), f(x_1), f'(x_0)$, $p(x) = f(x_0)(1-x^2) + f(x_1)x^2 + f'(x_0)(x-x^2)$

is quadratic and satisfies $p(0)=f(0)$, $p(1)=f(1)$, $p'(0)=f'(0)$

The required quadrature rule is therefore

$$\int_0^1 f(x)\,dx \simeq \int_0^1 p(x)\,dx = f(0)\int_0^1 (1-x^2)\,dx + f(1)\int_0^1 x^2\,dx + f'(0)\int_0^1 x-x^2\,dx$$

$$= \frac{2}{3}f(0) + \frac{1}{3}f(1) + \frac{1}{6}f'(0)$$

(unseen generalisation: Hermite interpolation is described in lectures and Newton-Cotes quadrature, but not this hybrid) [7 marks]

$$\int_{-\frac{\pi}{2}}^{0} \cos x\,dx \underset{(x=\frac{\pi}{2}t-\frac{\pi}{2})}{=} \frac{\pi}{2}\int_0^1 \cos\left(\frac{\pi}{2}t-\frac{\pi}{2}\right)dt$$

$$\simeq \frac{\pi}{2}\left[\frac{2}{3}\cos\left(-\frac{\pi}{2}\right) + \frac{1}{3}\cos(0) + \frac{1}{6}\frac{\pi}{2}\right] = \underline{\pi/6 + \pi^2/24}$$

quadrature rule

since $\frac{d}{dt}\left[\cos\left(\frac{\pi}{2}t-\frac{\pi}{2}\right)\right] = -\frac{\pi}{2}\sin\left(\frac{\pi}{2}t-\frac{\pi}{2}\right)$ so $f'(0) = \pi/2$

and $p\left(\frac{\pi}{2}t-\frac{\pi}{2}\right)$ on $[0,1]$ is $p(x)$ on $[-\frac{\pi}{2}, 0]$.

(Seen in the context of Gauss Quadrature, but not for this type of rule) [3 marks]

Part A Numerical Analysis 2016 Q2 Solution

$Q \in \mathbb{R}^{n\times n}$ is an orthogonal matrix if $Q^T = Q^{-1}$ [1 mark]

If $A \in \mathbb{R}^{n\times n}$, $A = QR$ is a QR factorization if $Q \in \mathbb{R}^{n\times n}$ is orthogonal and $R \in \mathbb{R}^{n\times n}$ is upper triangular. (Bookwork) [2 marks]

$Ax = b \Leftrightarrow QRx = b \Leftrightarrow Rx = Q^T b$ — (*) so find x by performing back substitution with the upper triangular matrix R in (*). Note A nonsingular $\Rightarrow Q^TA = R$ is nonsingular as orthogonal matrices are nonsingular by definition. [2 marks]

$A^2 y = c \Leftrightarrow QRQRy = c \Leftrightarrow RQRy = Q^Tc$

so solve $Rz = Q^Tc$ by back substitution,

then $QRy = z$ so solve $Ry = Q^Tz$ by back substitution. [2 marks]

(1st part are bookwork, 2nd simple extension similar to what is described in lectures for Gauss elim.)

$A = LU$, $L \in \mathbb{R}^{n\times n}$ is lower triangular with 1's on the diagonal and $U \in \mathbb{R}^{n\times n}$ is upper triangular. (Bookwork) [2 marks]

Performing Gauss Elimination on B

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & -4 \end{bmatrix} \to \begin{matrix} \\ \tfrac{1}{2} \\ \tfrac{1}{2} \end{matrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -\tfrac{1}{2} & \tfrac{3}{2} \\ 0 & \tfrac{1}{2} & -\tfrac{9}{2} \end{bmatrix} \to \begin{matrix} \\ \\ -1 \end{matrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -\tfrac{1}{2} & \tfrac{3}{2} \\ 0 & 0 & -3 \end{bmatrix}$$

so $L = \begin{bmatrix} 1 & 0 & 0 \\ \tfrac{1}{2} & 1 & 0 \\ \tfrac{1}{2} & -1 & 1 \end{bmatrix}$, $U = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -\tfrac{1}{2} & \tfrac{3}{2} \\ 0 & 0 & -3 \end{bmatrix}$

(simple example) [4 marks]

Solving $Ly = b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow$ $y_1 = 1$

$\tfrac{1}{2}y_1 + y_2 = 0 \Rightarrow y_2 = -\tfrac{1}{2}$

$\tfrac{1}{2}y_1 - 1(y_2) + y_3 = 0 \Rightarrow y_3 = -1$

and $Ux = y = \begin{bmatrix} 1 \\ -\tfrac{1}{2} \\ -1 \end{bmatrix}$ $\Rightarrow$ $2x_1 + (2) + (\tfrac{1}{3}) = 1 \Rightarrow x_1 = -\tfrac{2}{3}$

$-\tfrac{1}{2}x_2 + \tfrac{3}{2}(\tfrac{1}{3}) = -\tfrac{1}{2} \Rightarrow x_2 = 2$

$x_3 = \tfrac{1}{3}$

so soln $\begin{bmatrix} -\tfrac{2}{3} \\ 2 \\ \tfrac{1}{3} \end{bmatrix}$

(Standard unseen example) [3 marks]

3A Num. Anal. 2016 Q2 Soln Cont.

$A_1 \leftarrow A$

for $k = 1, 2, \ldots$

$A_k = L_k U_k$

$A_{k+1} = U_k L_k$

end.

so $A_{k+1} = L_k^{-1} L_k U_k L_k = L_k^{-1} A_k L_k$

hence A_{k+1} is similar to A_k and thus to $A_{k-1}, \ldots, A_1 = A$

(Unseen, but essentially same as QR alg which is Bookwork) [3 marks]

For $B = B_1 = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & -1 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & 1 & 1 \\ 0 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 & -3 \end{bmatrix}}_{U}$ we have $B_2 = UL = \begin{bmatrix} 3 & 0 & 1 \\ 1/2 & -2 & 3/2 \\ -3/2 & 3 & -3 \end{bmatrix}$ — (+)

And now apply Gerschgorin's Theorem:

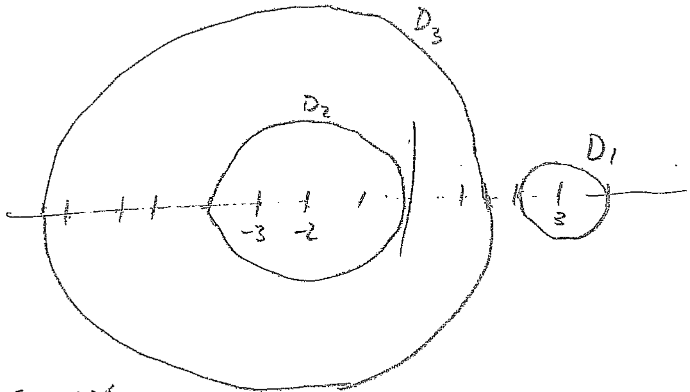
For any $A \in \mathbb{R}^{n \times n}$ every eigenvalue λ is contained in at least one of the discs $D_i = \{ z \in \mathbb{C} : |z - a_{ii}| \le \sum_{\substack{j=1 \\ j \ne i}}^{n} |a_{ij}| \}$

The discs for B_2 from (+) are

D_1 centre 3 radius 1

D_2 centre -2 radius 2

D_3 centre -3 radius $\frac{9}{2}$



Now D_1 is disjoint from $D_2 \cup D_3$ so we can apply Gerschgorin's 2nd Theorem:

If any set of k discs is disjoint from the remaining discs then it must contain exactly k eigenvalues

Thus D_1 contains 1 eigenvalue which must therefore be real since B_2 is real

Thus by the above B_2 is similar to B which must therefore have an eigenvalue in the interval $[2, 4]$

(Unseen use of Gerschgorin's Theorems which are Bookwork) [6 marks]

Part A Numerical Analysis 2016 Q3 Solution

$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx$ (*)

$P_0 = 1$ and $P_1 = x - \frac{\langle x,1\rangle}{\langle 1,1\rangle}1 = x$ as $\langle x,1\rangle = \int_{-1}^{1} x\,1\,dx = 0$.

$P_2 = x^2 - \frac{\langle x^2,x\rangle}{\langle x,x\rangle}x - \frac{\langle x^2,1\rangle}{\langle 1,1\rangle}1 = x^2 - \int_{-1}^{1} x^2 1\,dx \Big/ \int_{-1}^{1} 1\,1\,dx$

0 as $\int_{-1}^{1} x^2 x\,dx = 0$

$= x^2 - 2/3/2 = x^2 - \frac{1}{3}$.

$P_3 = x^3 - \frac{\langle x^3, x^2-\frac{1}{3}\rangle}{\langle x^2-\frac{1}{3}, x^2-1/3\rangle}(x^2-\frac{1}{3}) - \frac{\langle x^3,x\rangle}{\langle x,x\rangle}x - \frac{\langle x^3,1\rangle}{\langle 1,1\rangle}1$

0 as $\int_{-1}^{1} x^5\,dx = 0 = \int_{-1}^{1} x^3\,dx$. 0

$= x^3 - \int_{-1}^{1} x^3 x\,dx \Big/ \int_{-1}^{1} x\,x\,dx = x^3 - (2/5 / 2/3)x = x^3 - \frac{3}{5}x$

[A ; 6 marks]

Now since $P_k \in \Pi_k$, $\{P_0, P_1, \dots, P_k\}$ is a basis for Π_k so

$\langle P_k, q\rangle = 0 \quad \forall q \in \Pi_{k-1}$: Proof: write $q = \sum_{j=1}^{k-1} \alpha_j P_j$, $\alpha_j \in \mathbb{R}$

so $\langle P_k, q\rangle = \langle P_k, \sum_{j=1}^{k-1}\alpha_j P_j\rangle = \sum_{j=1}^{k-1}\alpha_j\langle P_k, P_j\rangle = 0$ as

by symmetry and linearity in $\langle\cdot,\cdot\rangle$

$\{P_0, P_1, \dots, P_k, \dots\}$ are the orthogonal polynomials w.r.t. the inner product (*).

So $0 = \int_{-1}^{1} P_k(x) P_0(x)\,dx = \int_{-1}^{1} P_k(x)\,dx$ so P_k changes sign at least once on $(-1,1)$. Assume it changes sign at l distinct points $r_1, \dots, r_l$ with $1 \le l \le k$ then let $q(x) = \prod_{i=1}^{l}(x - r_i)$ so that $P_k(x)q(x)$ does not change sign on $(-1,1)$ which contradicts $\langle P_k, q\rangle = 0$ as proved above unless $l = k$; hence P_k has k simple roots in $(-1,1)$

[B 6 marks]

$\|f\| = \sqrt{\langle f, f\rangle}$.

[B 1 mark]

Part A Numerical Analysis 2016 Q3 Soln Cont

If $W \subset V$ and $f \in V$, the $p \in W$ is a best approx if $\|f-p\| \le \|f-q\| \quad \forall q \in W$

[B : 1 mark]

If $\langle f-p, q\rangle = 0 \quad \forall q \in W$ then for any $r = p+q \in W$

$$\|f-r\|^2 = \langle f-p-q, f-p-q\rangle \underset{\substack{\text{linearity and}\\ \text{symmetry of } \langle\cdot,\cdot\rangle}}{=} \langle f-p, f-p\rangle - 2\underbrace{\cancel{\langle f-p, q\rangle}}_{0} + \langle q, q\rangle$$

$$\ge \langle f-p, f-p\rangle = \|f-p\|^2$$

so $\langle f-p, q\rangle = 0 \quad \forall q \in W \Rightarrow p \in W$ is a best approx to $f \in V$.

[B ; 3 marks]

$p = ax+b \in \Pi_1 = W$ is best approx to $x^3 \in \Pi_3 = V$ in $\circledast$ if

$$\int_{-1}^{1} (x^3 - ax - b)\, \frac{1}{x}\, dx = 0 \quad \text{i.e.} \quad \begin{array}{l} 0a + 2b = 0 \Rightarrow b = 0 \\ \frac{2}{3}a + 0b = \frac{2}{5} \Rightarrow a = \frac{3}{5} \end{array}$$

so $\underline{p(x) = \frac{3}{5}x}$.

[U, A 2 marks]

$A = M_n = \mathbb{R}^{n\times n}$, $\langle A, B\rangle = \mathrm{Trace}(B^T A) = \sum_{i=1}^{n}\sum_{k=1}^{n} b_{ki} a_{ki}$

so $\|A\|^2 = \langle A, A\rangle = \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ki}^2$ so $\|A\| = \left(\sum_{k=1}^{n}\sum_{i=1}^{n} a_{ki}^2\right)^{1/2}$

i.e. the square root of the sum of the squares of all of the entries

[U , 2 marks]

$D \in W$ means $D \in \mathbb{R}^{n\times n}$ is diagonal so for any $A \in M_n$

$$\langle A, D\rangle = \sum_{i=1}^{n}\sum_{k=1}^{n} d_{ki} a_{ki} = \sum_{i=1}^{n} d_{ii} a_{ii} \quad \text{as } d_{ki} = 0 \text{ if } k \ne i$$

Then $\langle A, D\rangle = 0$ when $a_{ii} = 0$ hence $\mathcal{D} = \mathrm{diag}(A)$ is the best approx to A from W since then $\langle A - \mathcal{D}, D\rangle = 0 \quad \forall D \in W$

[Frobenius inner product unseen : 4 marks]