A8 Probability – solutions

Question (1) [convergence, MGFs]

Marks: (a) 3+4, (b) 1+2+2, (c) 2+7+4.

(a) S (b) (i) B (ii) S (iii) S (c) (i) N (ii) S (iii) N.

Comments: (a) and (b) are standard. Part (c)(i)-(ii) is essentially a special case of the proof of the central limit theorem that was given in lectures. Part (iii) of (c) is new.

Solution (1):

(a) If $x \geq 1$ then $\mathbb{P}(M_n \leq x) = 1$ for all n.

If $x < 1$ then

$$\begin{aligned}\mathbb{P}(M_n \leq x) &= \mathbb{P}(X_i \leq x \text{ for } i = 1, 2, \dots, n)\\ &= \mathbb{P}(X_1 \leq x)^n \text{ since } X_i \text{ are i.i.d.}\\ &= \max(0, x)^n\\ &\to 0 \text{ as } n \to \infty.\end{aligned}$$

So indeed $F_{M_n}(x) \to F(x)$ for all x, where $F(x) = I(x \geq 1)$ is the c.d.f. of the constant r.v. 1.

Hence $M_n \to 1$ in distribution as $n \to \infty$, as required.

If $x \leq 0$ then $\mathbb{P}(n(1 - M_n) \leq x) = 0$. So let $x > 0$. Then

$$\begin{aligned}\mathbb{P}(n(1 - M_n) \leq x) &= \mathbb{P}(M_n \geq 1 - \frac{x}{n})\\ &= 1 - \mathbb{P}(M_n < 1 - \frac{x}{x})\\ &= 1 - \left(1 - \frac{x}{n}\right)^n \text{ for large enough } n\\ &\to 1 - e^{-x} \text{ as } n \to \infty.\end{aligned}$$

$I(x > 0)(1 - e^{-x})$ is the c.d.f. of an Exp(1) random variable. So we have $n(1 - M_n) \to \text{Exp}(1)$ in distribution as $n \to \infty$.

(b) The moment generating function of a random variable X is the function $M_X(t) = \mathbb{E}(e^{tX})$.

If $X \sim \text{Poisson}(\mu)$,

$$\begin{aligned} M_X(t) &= \sum_{n=0}^{\infty} e^{tn} \frac{e^{-\mu}\mu^n}{n!} \\ &= e^{-\mu} \sum_{n=0}^{\infty} \frac{(e^t\mu)^n}{n!} \\ &= e^{-\mu} \exp(e^t\mu) \\ &= \exp\big(\mu(e^t - 1)\big). \end{aligned}$$

If $X \sim N(0,1)$, then

$$\begin{aligned} M_X(t) &= \int_{-\infty}^{\infty} e^{xt} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \\ &= e^{t^2/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x-t)^2}{2}\right) dx \\ &= e^{t^2/2} \end{aligned}$$

since the integrand on the previous line is the p.d.f. of $N(t,1)$.

(c) (i) We have $N_k \sim \text{Poisson}(k)$, with mean k and variance k, so

$$Y_k = \frac{N_k}{\sqrt{k}} - \sqrt{k}.$$

Then

$$\begin{aligned} M_{Y_k}(t) &= \mathbb{E}\left(\exp\left(\frac{t}{\sqrt{k}} N_k - t\sqrt{k}\right)\right) \\ &= e^{-t\sqrt{k}} M_{N_k}(t/\sqrt{k}) \\ &= e^{-t\sqrt{k}} \exp\left(k\left(\exp(t/\sqrt{k}) - 1\right)\right) \end{aligned}$$

(from above, since $N_k \sim \text{Poisson}(k)$)

$$= \exp\left(-t\sqrt{k} + k[\exp(t/\sqrt{k}) - 1]\right).$$

(ii) Now consider $k \to \infty$. Then $t/\sqrt{k} \to 0$. So by the Taylor expansion of $\exp(x)$ around $x = 0$,

$$\exp(t/\sqrt{k}) - 1 = \frac{t}{\sqrt{k}} + \frac{t^2}{2k} + o\left(\frac{1}{k}\right).$$

Hence

$$\begin{aligned} M_{Y_k}(t) &= \exp\left(-t\sqrt{k} + t\sqrt{k} + \frac{t^2}{2} + o\left(\frac{1}{k}\right)\right) \\ &\to \exp\left(\frac{t^2}{2}\right) \quad \text{as } k \to \infty. \end{aligned}$$

This holds for all $t \in \mathbb{R}$.

Now $\exp(t^2/2)$ is the m.g.f. of $N(0,1)$, and is finite for all $t \in \mathbb{R}$.

Continuity theorem for m.g.fs.: suppose Y is a r.v. whose m.g.f. is finite on an open interval I containing the origin, and $M_{Y_k}(t) \to M_Y(t)$ as $k \to \infty$ for all $t \in I$; then $Y_k \to Y$ in distribution as $k \to \infty$.

So in this case $Y_k \to N(0,1)$ in distribution as $k \to \infty$.

(iii) Let

$$Z_k = \frac{N_{2k} - 2N_k}{\sqrt{k}}.$$

Then

$$Z_k = \frac{(N_{2k} - N_k) - \mathbb{E}(N_{2k} - N_k)}{\sqrt{\operatorname{Var}(N_{2k} - N_k)}} - \frac{N_k - \mathbb{E}N_k}{\sqrt{\operatorname{Var}(N_k)}}.$$

By stationary independent increments, N_k and $N_{2k} - N_k$ are i.i.d. So the two terms in the expression for Z_k above are i.i.d. and each has the same distribution as Y_k above.

Hence Z_k has m.g.f. $M_{Y_k}(t)M_{Y_k}(-t)$ which converges to e^{t^2} which is the m.g.f. of $N(0,2)$.

So $Z_k \to N(0,2)$ in distribution as $k \to \infty$.

An acceptable answer would also be to observe that both terms converge in distribution to $N(0,1)$, and the two terms are independent, so that the difference converges to the distribution of $Z - Z'$ where Z and Z' are i.i.d. $\sim N(0,1)$, which gives $N(0,2)$ (even without giving a precise result supporting this argument, which was not covered in the course).

Question (2) [transformation/change of variables]

Marks: (a) (6+2+2)+5; (b) 5+2+3.

(a): (i) S,S,N; (ii) S; (b) N.

Comments: (a) similar to questions on example sheets. (b) familiar techniques but a new example. The transformation is linear so the Jacobian is easy, but one needs to be particularly careful with the range on which the joint density is non-zero (experience suggests that students find this tricky). (b)(ii) is there partly to prompt a sanity-check on the answer to (b)(i).

Solution (2):

(a) (i) $f_{X,Y}(x,y) = \frac{1}{2\pi} \exp\left(-\frac{x^2+y^2}{2}\right)$.

If $x = r\cos\theta$ and $y = r\sin\theta$ then $x^2 + y^2 = r^2$.

We have

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial r} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = |r|.$$

So

$$\begin{aligned} f_{R,\Theta}(r,\theta) &= r f_{X,Y}(x,y) \\ &= \frac{1}{2\pi} r \exp\left(-\frac{r^2}{2}\right) \end{aligned}$$

for $\theta \in [0, 2\pi)$ and $r \geq 0$.

This density is a product, so R and Θ are independent. The density is constant in θ, so Θ is uniform on $[0, 2\pi)$. Then the density of R is $r\exp(-r^2/2)$ on $[0, \infty)$.

We have

$$\begin{aligned} \mathbb{P}(R^2 \leq u) &= \mathbb{P}(R \leq \sqrt{u}) \\ &= \int_0^{\sqrt{u}} r\exp(-r^2/2)dr \\ &= \left[-\exp(-r^2/2)\right]_0^{\sqrt{u}} \\ &= 1 - \exp(-u/2). \end{aligned}$$

So R^2 has exponential distribution with parameter 1/2.

In summary, Θ is uniform on $[0, 2\pi)$, and R^2 has exponential distribution with parameter 1/2, and they are independent.

(ii) Here one wants to draw a simple diagram, to obtain that

$$\begin{aligned}\mathbb{P}(Y > c|X|) &= \mathbb{P}\left(\Theta \in \left(\frac{\pi}{2} - \alpha, \frac{\pi}{2} + \alpha\right)\right) \quad \text{where } \alpha = \tan^{-1}\frac{1}{c} \\ &= \frac{2\alpha}{2\pi} \\ &= \frac{\tan^{-1}\frac{1}{c}}{\pi}.\end{aligned}$$

Then

$$\begin{aligned}\mathbb{P}\left(\left|\frac{Y}{X}\right| > c\right) &= \mathbb{P}(Y > c|X|) + \mathbb{P}(Y < -c|X|) \\ &= 2\mathbb{P}(Y > c|X|) \text{ by symmetry} \\ &= \frac{2\tan^{-1}\frac{1}{c}}{\pi}.\end{aligned}$$

So

$$\begin{aligned}F_{\left|\frac{Y}{X}\right|}(c) &= 1 - \frac{2\tan^{-1}\frac{1}{c}}{\pi} \\ &= \frac{2}{\pi}\left(\frac{\pi}{2} - \tan^{-1}\frac{1}{c}\right) \\ &= \frac{2}{\pi}\tan^{-1} c\end{aligned}$$

is the c.d.f. of $|Y/X|$.

(b) (i) Now $f_{X,Y}(x,y) = e^{-(x+y)}$ for $x, y > 0$.

If $u = x + y$ and $v = x + 2y$ then $x = 2u - v$ and $y = v - u$.

The Jacobian is given by

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} = 1.$$

So

$$f_{U,V}(u,v) = f_{X,Y}(x,y) = e^{-u}$$

on the set where $x, y > 0$, which corresponds to the set $u > 0$ and $u < v < 2u$ (and 0 elsewhere).

(ii) On the range where it is non-zero, the density function is constant in v. So given $U = u$, V is uniform on the interval $(u, 2u)$.

(iii)

$$f_V(v) = \int_{u=v/2}^{v} e^{-u} du = \left[-e^{-u}\right]_{v/2}^{v} = e^{-v/2} - e^{-v}.$$

Question (3) [Markov chain]

Marks: (a) 2 (b)(i) 5 (ii) 3 (iii) 6 (c)(i) 4 (ii) 5 (though I would be prepared to mark in particular part (c) flexibly to give suitable credit for partial answers.

(a) B (b) S (c)(i) N (ii) N.

The simple random walk has been seen in many different guises during the course. The calculation of hitting probabilities in (b) was done in lectures. The questions in part (c) have not been seen before and could be more testing, although not completely unfamiliar – calculations like (ii) came up in the context of convergence to equilibrium / ergodic theorems.

Solution (3):

(a) Let $X_i, i \geq 0$ be a sequence of i.i.d. random variables with mean μ. Then

$$\mathbb{P}\left(\frac{\sum_{i=1}^n X_i}{n} \to \mu \text{ as } n \to \infty\right) = 1.$$

(b) (i) We can write $Y_n = Y_0 + \sum_{i=1}^n X_i$, where X_i are i.i.d. with $\mathbb{P}(X_i = 1) = p$, $\mathbb{P}(X_i = -1) = 1 - p$. So $\mathbb{E}X_i = p - (1-p) = 2p - 1, = \mu$, say.

Hence by SLLN, $\sum_{i=1}^n X_i/n \to \mu$ with probability 1, as $n \to \infty$.

But $Y_n/n = k/n + \sum_{i=1}^n X_i/n$. Since $k/n \to 0$ as $n \to \infty$, we have $Y_n/n \to \mu$ as $n \to \infty$ with probability 1, as desired.

(ii) Suppose $Y_0 \leq 0$ and $Y_n \to \infty$ as $n \to \infty$. Then, since Y only moves up by 1 step at a time, Y_n must be 0 for some n. From (i) we know $Y_n \to \infty$ as $n \to \infty$ with probability 1.

So if $k \leq 0$ then $h_k = 1$.

[One could equally well do this using minimal non-negative solutions to recurrence relations as in (iii) below]

(iii) The hitting probabilities satisfy

$$h_i = \sum_j p_{ij} h_j \text{ for } i \neq 0$$
$$h_0 = 1.$$

In fact, (h_i) is the minimal non-negative solution to this system of equations.

In this case we have $h_i = p h_{i+1} + (1-p) h_{i-1}$.

To solve this consider the auxiliary equation $x = px^2 + (1-p)$, giving $(x - 1)(px - (1-p)) = 0$ and so $x = 1$ or $x = (1-p)/p$.

So general solutions are

$$h_i = A + B\left(\frac{1-p}{p}\right)^i.$$

We want the minimal solution such that $h_0 = 1$, i.e. such that $A + B = 1$. This is given by $A = 0, B = 1$. So $h_i = \beta^i$ for $\beta = (1-p)/p$, as required.

(c) (i) From 0, the probability of visiting $-k$ is the same as from k, the probability of visiting 0, i.e. h_k.

Since the chain only moves down one step at a time, if the chain ever goes below $-k$, it must visit $-k$ at some point. So

$$\begin{aligned}\mathbb{P}(W \leq -k) &= h_k \\ &= \left(\frac{1-p}{p}\right)^k.\end{aligned}$$

So

$$\mathbb{P}(-W \geq k) = \left(\frac{1-p}{p}\right)^k.$$

That is, $-W$ is geometric (starting from 0) with parameter $1 - \frac{1-p}{p} = \frac{2p-1}{p}$.

(ii) Let $\mu = 2p - 1$.

We have $T_j \to \infty$ as $j \to \infty$.

So if it's true that $Y_n/n \to \mu$ as $n \to \infty$, then also $Y_{T_j}/T_j \to \mu$ as $j \to \infty$.

But $Y_{T_j} = j$, so this means $j/T_j \to \mu$, which means $T_j/j \to 1/\mu$.

Because of (a)(i), we therefore have $\mathbb{P}(T_j/j \to \alpha) = 1$ as required, where $\alpha = 1/\mu = 1/(2p-1)$.