1. (a) *2+2+5+2+3 marks for standard but unseen questions, similar to questions on problem sheets.*

   (i) $\int_x^\infty f(x,y)dy = \int_0^\infty cxze^{-x-z}dz = cxe^{-x}$ recognising the integral for the mean of a standard exponential distribution. Similarly, integrating over $x \in (0,\infty)$, we obtain integral c, so we need $c = 1$.

   (ii) X and Y are not independent since the probability density function does not factorise into a function of x and a function of y.

   (iii) From (i), $f_X(x) = \int_x^\infty f(x,y)dy = xe^{-x}$, $x > 0$. So, $X \sim \text{Gamma}(2,1)$. Similarly, $f_Y(x) = \int_0^y x(y-x)e^{-y}dy = e^{-y}[\frac{1}{2}x^2y - \frac{1}{3}x^3]_0^y = \frac{1}{6}y^3e^{-y}$, $y > 0$, so Y is $\text{Gamma}(4,1)$. Hence, as sums of independent exponential variables with mean and variance 1, we have $\mathbb{E}(X) = \text{Var}(X) = 2$, $\mathbb{E}(Y) = \text{Var}(Y) = 4$, by independence.

   (iv) $f_{X|Y=y}(x) = f(x,y)/f_Y(y) = 6x(y-x)/y^3$, $x \in (0,y)$.

   (v) $\mathbb{E}(XY) = \int_0^\infty \int_0^y x^2y(y-x)e^{-y}dxdy = \int_0^\infty (\frac{1}{3}y^5 - \frac{1}{4}y^5)e^{-y}dy = 10$ integral of $\text{Gamma}(6,1)$ density, noting that $\Gamma(6) = 5! = 120$. Hence $\text{Cov}(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = 2$. Students may also represent $Y = X + X'$ for $X' \sim \text{Gamma}(2,1)$ independent of X and use this to calculate $\text{Cov}(X,Y)$.

   (b) *(i) 3 marks for bookwork, (ii) 4 marks for standard application, (iii) 4 marks for something new.*

   (i) Let $(X_j, j \geq 1)$ be a sequence of independent identically distributed random variables with mean μ and variance $\sigma^2 \in (0,\infty)$. Let $S_n = X_1 + \cdots + X_n$, $n \geq 1$. Then

$$\mathbb{P}\left(\frac{S_n - n\mu}{\sqrt{n\sigma^2}} \leq x\right) \to \int_{-\infty}^x \frac{1}{\sqrt{2\pi}}e^{-z^2/2}dz = \mathbb{P}(Z \leq x).$$

   (ii) The Central Limit Theorem applies since X and Y have positive finite variance, by (a)(iii). By symmetry $\mathbb{P}(Z \leq 0) = 1/2$. Hence

$$\mathbb{P}(S_n \leq 2n) = \mathbb{P}\left(\frac{S_n - n\mathbb{E}(X)}{\sqrt{n\text{Var}(X)}} \leq 0\right) \to \frac{1}{2}$$
$$\mathbb{P}(T_n \leq 4n) = \mathbb{P}\left(\frac{T_n - n\mathbb{E}(Y)}{\sqrt{n\text{Var}(Y)}} \leq 0\right) \to \frac{1}{2}.$$

   (iii) S_n and T_n are not independent, but writing $T_n = S_n + S_n'$ for independent S_n and S_n', we have that $(S_n - 2n)/\sqrt{2n}$ and $(S_n' - 2n)/\sqrt{2n}$ converge in distribution to independent normals Z and Z', say, and hence

$$\mathbb{P}(S_n \leq 2n, T_n \leq 4n) = \mathbb{P}\left(\frac{S_n - 2n}{\sqrt{2n}} \leq 0, \frac{S_n - 2n}{\sqrt{2n}} + \frac{S_n' - 2n}{\sqrt{2n}} \leq 0\right)$$
$$\to \mathbb{P}(Z < 0, Z + Z' < 0) = \mathbb{P}\left(\Theta \in \left(\pi, \frac{7}{4}\pi\right)\right) = 3/8,$$

   by spherical symmetry of the standard bivariate normal distribution, which makes the angular part $\Theta \sim \text{Unif}(0, 2\pi)$.

2. (a) *3 marks for bookwork.*

Let $(X_j, j \geq 1)$ be a sequence of independent identically distributed random variables with finite mean μ and $S_n = X_1 + \cdots + X_n$, $n \geq 1$. Then $\mathbb{P}(S_n/n \to \mu) = 1$.

(b) *2+3+5 marks for a seen example, (i)-(ii) in lectures, (iii) on problem sheet.*

(i) For $p \in [1/2, 1)$, the only communicating class is $\mathbb{Z}$, while for $p = 1$, all $\{n\}$, $n \in \mathbb{Z}$ are singleton classes.

(ii) We can write $S_n = a + X_1 + \cdots + X_n$ for $\mathbb{P}(X_j = 1) = 1 - \mathbb{P}(X_j = -1) = p$, with $\mathbb{E}(X_j) = 2p - 1$. For $p > 1/2$, SLLN yields $(S_n - a)/n \to p > 1/2$ hence $S_n \to \infty$ a.s. For $p = 1/2$ the SLLN does not determine the limiting behaviour.

(iii) By translation invariance we may assume $a = 0$. Let

$$h_m = \mathbb{P}_m(S_n = 0 \text{ for some } n \geq 0), \qquad m \in \mathbb{Z}.$$

Then $(h_m, m \in \mathbb{Z})$ is the minimal solution of $h_0 = 1$, $h_m = ph_{m+1} + (1-p)h_{m-1}$. Let $p = 1/2$. Then, $h_1 < 1$ would imply $h_{-1} > 1$, which is absurd, so $h_1 = 1$. Inductively, $h_m = 1$ for all $m \in \mathbb{Z}$.
Let $p > 1/2$. Then, $h_m = 1$ for all $m < 0$ by (ii). For $m > 0$, try $h_m = \beta^m$. Then $\beta = p\beta^2 + (1-p)$, factorising $\beta^2 - \beta/p + (1-p)/p = (\beta - 1)(\beta - (1-p)/p)$, we see that $\beta = (1-p)/p$ yields the minimal solution $h_m = ((1-p)/p)^m$, $m \geq 0$. Conditioning on the first step and applying the Markov property, we find

$$h = ph_1 + (1-p)h_{-1} = 1 - p + 1 - p = 2 - 2p.$$

(c) *7 marks for a more challenging unseen application.*

(i) By irreducibility, we only need to check that 0 is recurrent. By symmetry, (b)(iii) for $p \in (0, 1/2]$, yields $h_m = 1$ for all $m \geq 1$. Here, conditioning on the first step and applying the Markov property, we find that the return probability to 0 is $h' = h_1 = 1$, so 0 is recurrent.

(ii) $\xi Q = \xi$ is equivalent to $\xi_m = \xi_{m-1}p + \xi_{m+1}(1-p)$. Try $\xi_m = A\gamma^m$, then, as above $\gamma \in \{p/(1-p), 1\}$. Only $h_m = A(p/(1-p))^m$ can be normalised, and only for $p < 1/2$, to satisfy $\sum_{m \geq 0} \xi_m = 1$, and $A = 1 - p/(1-p)$ yields a geometric distribution with parameter $A = (1-2p)/(1-p)$. For $p = 1/2$ there is no solution satisfying $\sum_{m \geq 0} \xi_m = 1$, so there is no stationary distribution.

(d) *5 marks for something new.*

In (c)(ii), the mean return time to 0 is $1/\xi_0 = (1-p)/(1-2p)$. For (d), denote the stationary distribution by η. Conditioning on the first step, the mean return time to 0 here is $m_0 = 1 + \frac{1}{2}(a^+ + a^-)$ where $a^+ = (1-p^+)/(1-2p^+) - 1 = p^+/(1-2p^+)$ and $a^- = p^-/(2p^- - 1) - 1 = (1-p^-)/(2p^- - 1)$, and $\eta_0 = 1/m_0$. By the Ergodic Theorem, this is also the long-term proportion of time spent in 0. Of the time not spent at 0, a proportion $a^+/(a^+ + a^-)$ is spent positive since each return from 1 takes a^+ on average, each return from -1 takes a^-. By (c)(ii), these proportions further split into proportions $\xi_m/(1-\xi_0)$ spent in m. Hence

$$\eta_m = (1-\eta_0)\frac{a^+}{a^+ + a^-}\frac{1-2p^+}{1-p^+}\left(\frac{p^+}{1-p^+}\right)^{m-1} = \frac{1}{2 + a^+ + a^-}\left(\frac{p^+}{1-p^+}\right)^m$$

for $m \geq 1$, and by symmetry, $\eta_m = (1/(2 + a^+ + a^-))((1-p^-)/p^-)^{|m|}$ for $m \leq -1$.

3. (a) *8=4+4 marks for a standard question, seen before.*

(i) $M(t) = \int_0^\infty e^{tx} f(x)dx = \int_0^\infty \frac{1}{\Gamma(r)} \lambda^r x^{r-1} e^{-(\lambda-t)x} dx = \left(\frac{\lambda}{\lambda - t}\right)^r$, for $t < \lambda$, as the Gamma$(r, \lambda - t)$ density integrates to 1.

(ii) $\text{Exp}(\lambda) = \text{Gamma}(1, \lambda)$. By independence, for all $t < \lambda$,

$$\mathbb{E}(e^{t(E_1+\cdots+E_n)}) = \mathbb{E}(e^{tE_1}) \cdots \mathbb{E}(e^{tE_n}) = \left(\frac{\lambda}{\lambda - t}\right)^n.$$

By the uniqueness theorem for moment generating functions, $E_1 + \cdots + E_n \sim \text{Gamma}(n, \lambda)$.

(b) *7=3+4 marks for bookwork.*

(i) $N_0 = 0$,
increments $N_{t_j} - N_{t_{j-1}}$, $1 \le j \le n$, are independent for all $0 = t_0 < t_1 < \cdots < t_n$, and $N_{s+t} - N_s \sim \text{Poisson}(\lambda t)$ for all $s, t \geq 0$.

(ii) By definition, for $s = 0$, we have $N_t = N_{0+t} - N_0 \sim \text{Poisson}(\lambda t)$.
$\mathbb{P}(T_n > t) = \mathbb{P}(N_t \le n - 1) = \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!} e^{-\lambda t}$. Hence, differentiation yields

$$f_{T_n}(t) = -\frac{d}{dt}\mathbb{P}(T_n > t) = \sum_{k=0}^{n-1} \left(\frac{\lambda^{k+1}t^k}{k!} - \frac{k\lambda^k t^{k-1}}{k!}\right) e^{-\lambda t} = \frac{1}{\Gamma(n)} \lambda^n t^{n-1} e^{-\lambda t},$$

for all $t > 0$. Hence, $T_n \sim \text{Gamma}(n, \lambda)$.

(c) *3 marks for bookwork.*

Let $(M_t, t \geq 0)$ and $(N_t, t \geq 0)$ be two independent Poisson processes of rates μ and λ. Then $K_t = M_t + N_t$, $t \geq 0$, is a Poisson process of rate $\mu + \lambda$.

(d) *8=2+4+2 marks for something new.*

(i) $\mathbb{P}(T_1 > t) = \mathbb{P}(L_t - L_0 = 0) = \exp(-\alpha t^m)$.
Hence $f_{T_1}(t) = \alpha m t^{m-1} \exp(-\alpha t^m)$, $t > 0$.

(ii) First note that $N_u - N_t \sim \text{Poisson}(\alpha(u^m - t^m))$, by independence of increments.

$$\begin{aligned}\mathbb{P}(S_1 \le t, S_2 > u) &= \mathbb{P}(L_t = 1, L_u - L_t = 0) = \alpha t^m e^{-\alpha t^m} e^{-\alpha(u^m - t^m)} \\ &= \alpha t^m e^{-\alpha u^m},\end{aligned}$$

for $0 < t < u$.
By differentiation, $f_{S_1,S_2}(t, u) = \alpha m t^{m-1} \alpha m u^{m-1} e^{-\alpha u^m}$, $0 < t < u$.
By the transformation formula, $f_{S_1,S_2-S_1}(t, s) = \alpha^2 m^2 t^{m-1} (s+t)^{m-1} e^{-\alpha(s+t)^m}$, $t > 0$, $s > 0$.
As $m > 1$, this does not factorise, so S_1 and $S_2 - S_1$ are not independent, in contrast to the case $m = 1$ of a (homogeneous) Poisson process.

(iii) $J_t = L_t + N_t$, $t \geq 0$, is an inhomogeneous Poisson process with rate function $\alpha m t^{m-1} + \lambda$. This is because independent increments are preserved as for the standard superposition theorem. Clearly $J_0 = 0$, and the rate function, which is the derivative of the parameter of the Poisson distribution of J_t is identified from $J_t = L_t + N_t \sim \text{Poisson}(\alpha t^m + \lambda t)$.