G.1

(a) To prove the Beltrami identity, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}(y'\frac{\partial F}{\partial y'} - F)$$

$$= y''\frac{\partial F}{\partial y'} + y'\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y'} - \frac{\partial F}{\partial x} - y'\frac{\partial F}{\partial y} - y''\frac{\partial F}{\partial y'}$$

which vanishes by cancellation of the first and last terms, by $\frac{\partial F}{\partial x} = 0$ given and by the E-L equation as given.

[5 marks]

This is standard bookwork for the course.

(b) In the example, the constant is

$$\frac{y'^2}{y\sqrt{1+y'^2}} - \frac{\sqrt{1+y'^2}}{y}$$

so that simplifying,

$$\frac{1}{y\sqrt{1+y'^2}} = k\,.$$

[3 marks]

This F and its properties have been used several times for the course.

To solve this DE by separating the variables, write as

$$k^2(1+y'^2) = y^{-2}$$

$$k^2y'^2 = y^{-2} - k^2$$

$$\pm\frac{k\,\mathrm{d}y}{\sqrt{y^{-2}-k^2}} = \mathrm{d}x$$

$$\pm\frac{k^2\,y\,\mathrm{d}y}{\sqrt{1-k^2y^2}} = k\mathrm{d}x$$

Integrating

$$\pm\sqrt{1-k^2y^2} = k(x-L)$$

so

$$(x-L)^2 + y^2 = k^{-2}$$

which is the equation of a circle with centre $(L, 0)$.

[6 marks for arriving at the equation of a circle by one means or another]

To find L in terms of the given boundary conditions, solve

$$(-1 - L)^2 + 4c^2 = (1 - L)^2 + 4d^2$$

$$4L = 4(d^2 - c^2)$$

as required.

[3 marks]

(c) The *natural boundary condition* for this problem is that $\frac{\partial F}{\partial y'} = 0$, i.e. $y' = 0$.

[3 marks]

This immediately fixes $L = 1$, and the solution satisfying $y(-1) = 2c$ is given by $(x - 1)^2 + y^2 = 4 + 4c^2$.

[3 marks]

This example is slightly different from those used in the course. A geometrical method of fixing the circle would also be acceptable.

If the significance of the condition $y' = 0$ is is understood geometrically, it is obvious that a circle cannot satisfy it at two values of x, and so there are no solutions satisfying natural boundary conditions at both $x = -1$ and $x = 1$.

Alternatively: observe that there are geodesics of a length that can be made as small as desired by going to larger y, and the infimum 0 cannot be attained.

[2 marks]