Solutions for Part A Graph Theory 2015

(a) (i) (2 marks) A tree is a connected acyclic graph. A leaf is a vertex with degree 1. BW

(3 marks) Let T be a tree with at least two vertices. Consider a maximal path P in T, which must have distinct end vertices u and v. u has exactly one neighbour on P (otherwise there would be a cycle), and no neighbour outside P (otherwise we could extend P), so u is a leaf. Similarly v is a leaf. BW

(ii) (2 marks) Any cycle in $T' = T \setminus v$ would be a cycle in T, so T' is acyclic. Let u and w be vertices in T'. Let P be a path between u and w in T. Then v is not in P, since otherwise the neighbour of v would have to appear twice on P. Thus P is a path in T', so T' is connected.

(iii) (7 marks) This is from problem set 1

Use induction on the number n of vertices in T. The result is trivial if $n = 1$: assume $n \geq 2$ and the result holds for trees with fewer vertices. Let v be a leaf of T, and let w be its unique neighbour. If some subtree T_i consists just of v, then each subtree must contain v and we are done: so suppose not.

Consider the tree $T' = T \setminus v$ and its (non-empty) subtrees $T_i' = T_i \setminus v$ for $i = 1, \ldots, k$. For $1 \leq i < j \leq k$, if T_i and T_j have the leaf v in common they must also have w in common, and so T_i' and T_j' must always have a vertex in common. Hence, by the induction hypothesis there is a vertex x in each T_i', and then x is in each T_i.

(b) (i) (3 marks) A matching is a set of pairwise disjoint edges (with no common vertices). A cover is a set of vertices which meets each edge in G. König's Theorem states that, in a bipartite graph, the maximum size of a matching equals the minimum size of a cover. BW

(ii) (4 marks) Let M be a maximum matching and let K be a minimum cover. By König's Theorem, $|M| = |K|$. But K must contain at least one end-vertex of each edge in M (since it is a cover), and if K contained both ends of some edge it would have size $> |M|$. So K must contain exactly one end-vertex of each edge in M, as required. BW

(iii) (4 marks) Note first that, in any graph, the maximum size of a matching is at most the minimum size of a cover. For if M is a matching and K a cover, then K contains at least one end vertex of each edge in M, so $|M| \leq |K|$. new

But by the result in (ii), there is a matching M and cover K with $|M| = |K|$, so the maximum size of a matching equals the minimum size of a cover, which is König's Theorem.