Group Theory Solution:

Solution: (a) [B] [3 marks] Using the relation $yx = x^4y$, any xs in a word can be brought to the left of any ys to be seen to be equivalent to a word x^iy^j. As x has order 9 and y has order 3 then every word can be put in the form of $(*)$.

[B] [2 marks] As $yx = x^4y$ then $yx^r = x^{4r}y$ and hence $y^jx^r = x^{r4^j}y^j$ so that

$$x^iy^jx^ry^s = x^{i+r4^j}y^{j+s}.$$

(b) [B/S] [3 marks] The underlying set of $N \rtimes_\varphi H$ is the product $N \times H$. The binary operator $\bullet$ is defined by

$$(n_1, h_1) \bullet (n_2, h_2) = (n_1\varphi(h_1)(n_2), h_1h_2).$$

Closure is clear. [1 mark] Note that we also have an obvious identity

$$(e_N, e_H) \bullet (n, h) = (n, h) = (n, h) \bullet (e_N, e_H),$$

as $\varphi(e_H) = id_N$. [2 marks] For inverses note [2 marks]

$$\begin{aligned} (\varphi(h^{-1})(n^{-1}), h^{-1}) \bullet (n, h) &= (\varphi(h^{-1})(n^{-1})\varphi(h^{-1})(n), e_H) = (\varphi(h^{-1})(e_N), e_H) = (e_N, e_H). \\ (n, h) \bullet (\varphi(h^{-1})(n^{-1}), h^{-1}) &= (n\varphi(h)\varphi(h^{-1})(n^{-1}), hh^{-1}) = (nn^{-1}, hh^{-1}) = (e_N, e_H). \end{aligned}$$

[2 marks] Finally we have that

$$\begin{aligned} (n_1, h_1) \bullet \{(n_2, h_2) \bullet (n_3, h_3)\} &= (n_1, h_1) \bullet (n_2\varphi(h_2)(n_3), h_2h_3) \\ &= (n_1\varphi(h_1)\left(n_2\varphi(h_2)(n_3)\right), h_1h_2) \\ &= (n_1\varphi(h_1)\left(n_2\right)\varphi(h_1h_2)(n_3), h_1h_2) \\ &= \{(n_1, h_1) \bullet (n_2, h_2)\} \bullet (n_3, h_3). \end{aligned}$$

[3 marks] We see that $G = C_9 \rtimes_\varphi C_3 \cong \langle x \rangle \rtimes_\varphi \langle y \rangle$ where $\varphi : C_3 \to \mathrm{Aut}(C_9)$ is the map

$$\varphi(y^j)(x^r) = (x^r)^{4^j}.$$

Note that $\varphi(y^j)(x^r)\varphi(y^j)(x^s) = \varphi(y^j)(x^{r+s})$ and that $\varphi(y^j)$ is an automorphism as 4^j is coprime with 9. Further φ is a homomorphism as

$$\varphi(y^j)\left(\varphi(y^k)(x^r)\right) = \left((x^r)^{4^j}\right)^{4^k} = x^{r(4^j4^k)} = x^{r(4^{j+k})} = \varphi(y^{j+k})\left(x^r\right).$$

(c) [S/N] [4 marks] As G is not abelian then the commutator subroup of G is non-trivial. However the calculations in (a)(ii) we can see that a commutator $a^{-1}b^{-1}ab$ will lie in $\langle x \rangle$ as the powers of y present in a^{-1}, b^{-1}, a, b will cancel out in the calucation (or equally from (b) we know $\langle x \rangle$ to be normal in G). Hence G' lies in $\langle x \rangle$ which abelian and so $(G')' = \{e\}$. As the derived series terminates then G is solvable and as $G' \neq \{e\}$ then the derived length is 2.

(d) [N] [3 marks] Note that

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}^3 = \begin{pmatrix} 1 & 2a & ac+2b \\ 0 & 1 & 2c \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3a & 3ac+3b \\ 0 & 1 & 3c \\ 0 & 0 & 1 \end{pmatrix} = I_3.$$

Hence the non-trivial elements of H have order 3 whilst there are elements of G (such as x) with order 9. The two groups are not isomorphic.