Integral Transforms Solution:

(a) [B] The Laplace transform of:
(i) $f'(x)$ equals $p\overline{f}(p) - f(0)$. [2 marks]
(ii) $\sin x$ equals $1/(p^2+1)$. [2 marks]
(iii) $\delta(x-a)$ equals e^{-ap}. [2 marks]

(b) [S] (i) Transforming the differential equation we find

$$\{p^2\overline{f}(p) - pf(0) - f'(0)\} + \overline{f}(p) = \overline{k}(p).$$

But $f(0) = f'(0) = 0$ and so

$$\overline{f}(p) = \frac{\overline{k}(p)}{p^2+1}. \qquad \text{[3 marks]}$$

The convolution of two functions $g * h$ is

$$(g*h)(x) = \int_0^x g(t)h(x-t)\,\mathrm{d}t$$

and $\overline{g*h} = \overline{g}\overline{h}$. [2 marks] Hence

$$f(x) = \int_0^x k(t)\sin(x-t)\,\mathrm{d}t \quad \text{or} \quad \int_0^x k(x-t)\sin(t)\,\mathrm{d}t. \qquad \text{[1 mark]}$$

(ii) When $k(x) = \delta(x-a)$ then

$$f(x) = \int_0^x \delta(t-a)\sin(x-t)\,\mathrm{d}t.$$

By the sifting property of the delta function

$$\int_{-\infty}^{\infty} g(t)\delta(t-a)\,\mathrm{d}t = g(a), \qquad \text{[1 mark]}$$

and separating the cases as to whether a is in the interval $(0,x)$ or not,

$$f(x) = \begin{cases} 0 & x < a; \\ \sin(x-a) & x \geqslant a. \end{cases} \qquad \text{[2 marks]}$$

(c) [S/N] Transforming the differential equation, using $\overline{xf}(p) = -\mathrm{d}\overline{f}/\mathrm{d}p$ [1 mark] and noting $J_0(0) = 1$, $J_0'(0) = 0$, we have

$$-\frac{\mathrm{d}}{\mathrm{d}p}\left(p^2\overline{J_0} - p\right) + \left(p\overline{J_0} - 1\right) - \frac{\mathrm{d}\overline{J_0}}{\mathrm{d}p} = 0.$$

$$\implies -2p\overline{J_0} - p^2\frac{\mathrm{d}\overline{J_0}}{\mathrm{d}p} + p\overline{J_0} - \frac{\mathrm{d}\overline{J_0}}{\mathrm{d}p} = 0.$$

$$\implies \frac{\mathrm{d}\overline{f}}{\mathrm{d}p} + \frac{p}{(p^2+1)}\overline{f} = 0.$$

This is separable and we see

$$\overline{J_0}(p) = \frac{A}{\sqrt{p^2+1}} \qquad \text{[4 marks]}$$

for some A. [N] However as

$$\overline{J_0'}(p) = p\overline{J_0}(p) - J_0(0) = \frac{Ap}{\sqrt{p^2+1}} - 1 \to 0 \quad \text{as } p \to \infty,$$

then $A = 1$. [2 marks]

[N] If we substitute $k(x) = J_0(x)$ into (b) then we have the inverse problem of

$$\overline{f}(p) = \frac{1}{(p^2+1)^{3/2}}.$$

The given function $f(x)$ equals the convolution of 1 and $xJ_0(x)$. Hence its Laplace Transform is

$$\frac{1}{p} \times -\frac{\mathrm{d}}{\mathrm{d}p}\left((p^2+1)^{-1/2}\right) = \frac{1}{p} \times -\left(\frac{-1}{2}2p(p^2+1)^{-3/2}\right) = (p^2+1)^{-3/2}$$

as required. [2 marks] As the Laplace transform is injective (or similar mention of the Laplace inverse) the result follows. [1 mark]