# Introduction to Manifolds

(a), (b) are Bookwork

(c), (d) are New (though the related AM/GM inequality is in the lecture notes online)

(a).

Let $g : \mathbb{R}^n \to \mathbb{R}^k$. We say $g$ is differentiable at $x$ if $\exists$ a linear map $dg_x : \mathbb{R}^n \to \mathbb{R}^k$ such that

$$g(x+h) = g(x) + dg_x(h) + R_x(h)$$

where $\lim_{h\to 0} \frac{\|R_x(h)\|}{\|h\|} = 0$

If $\operatorname{rank} dg_x = k \quad \forall x \in g^{-1}(0)$ then $g^{-1}(0)$ is a submanifold of $\mathbb{R}^n$ of dimension $n-k$.

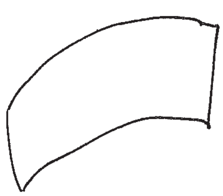
The tangent space to $g^{-1}(0)$ at $x$ is

$$T_x\, g^{-1}(0) = \ker dg_x$$

which has dim $n-k$.

(b). We need to find critical points of $f: \mathbb{R}^n \to \mathbb{R}$ on the constraint space $g^{-1}(0)$ where $g: \mathbb{R}^n \to \mathbb{R}^k$

We assume $g^{-1}(0)$ is a submanifold (assume the sufficient condition in (a)).



We need the component of $\nabla f$ tangent to $g^{-1}(0)$ to vanish.

Now $T_x g^{-1}(0) = \ker dg_x$

$$= \operatorname{Span}\{\nabla g_1(x), \dots, \nabla g_k(x)\}^{\perp}$$

where $g = (g_1, \dots, g_k)$

so the normal space to $g^{-1}(0)$ is $\operatorname{Span}\{\nabla g_1(x), \dots, \nabla g_k(x)\}$

Our eqn. for a critical point is therefore

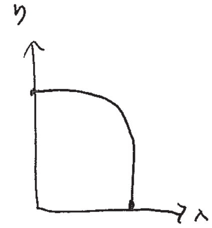
$$\nabla f = \sum_{i=1}^{k} \lambda_i \nabla g_i$$ where $\lambda_i$ are free scalars

together with constraint $g = 0$.

(c). Consider $f(x,y) = xy$

on $x^p/p + y^q/q = 1$ ; $x, y \geq 0$

where $\frac{1}{p} + \frac{1}{q} = 1$.




Clearly the min of $f$ is attained at the endpoints on the axes.

By compactness the max must also be attained, at a point where $\nabla f = \lambda \nabla g$

$\left( g = x^p/p + y^q/q - 1 \right)$

This eqn. is

$$\begin{pmatrix} y \\ x \end{pmatrix} = \lambda \begin{pmatrix} x^{p-1} \\ y^{q-1} \end{pmatrix}$$

note $\nabla g$ is never $0$ in $g^{-1}(0)$

so $y = \lambda x^{p-1}$

now $\lambda y^{q-1} = \lambda^q x^{(p-1)(q-1)}$ and this must $= x$

but $\frac{1}{p} + \frac{1}{q} = 1 \iff p + q = pq$

$\iff (p-1)(q-1) = 1$ so $(p-1)q = p$

so $\lambda y^{q-1} = \lambda^q x$

so we get a solution if $\lambda^q = 1$.

Now $x, y > 0$ so $\lambda > 0$ so $\lambda = 1$

$(\in \mathbb{R})$

Our soln. is therefore

$$y = x^{p-1}$$

now $1 = x^p/p + y^q/q = x^p/p + \frac{x^{(p-1)q}}{q}$

$= \frac{x^p}{p} + \frac{x^p}{q}$ as $(p-1)q = p$

$= x^p$

so $\underline{x = 1, y = 1.}$ and the max value of $f = xy$ is $1$.

(d) let $r = x^p/p + y^q/q$ and

define $x' = \frac{x}{r^{1/p}}$ ; $y' = \frac{y}{r^{1/q}}$

so $\frac{(x')^p}{p} + \frac{(y')^q}{q} = \frac{x^p}{pr} + \frac{y^q}{qr} = 1$

The above result in (c) $\Rightarrow x'y' \leq 1$

$\Rightarrow \frac{xy}{r} \leq 1$

$\Rightarrow xy \leq r$ which is the desired result.