Projective Geometry

(a), (b) are Bookwork

(d) and first 2 parts of (e) are Seen

(c) and last part of (e) are New.

(a) Let $P_i = [x_i, y_i]$ $i = 0, 1, 2, 3$

cross-ratio $(P_0 P_1 : P_2 P_3)$ is

$$\frac{(x_0 y_2 - x_2 y_0)(x_1 y_3 - x_3 y_1)}{(x_0 y_3 - x_3 y_0)(x_1 y_2 - x_2 y_1)}$$

$$\begin{pmatrix} x_i & x_j \\ y_i & y_j \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_i & x_j \\ y_i & y_j \end{pmatrix}$$ under projective transformation

so each $x_i y_j - x_j y_i$ scales by $\det = ad - bc \neq 0$

and cross-ratio is invariant.

(b).

Thm Any 3 distinct points in $\mathbb{CP}^1$ may be moved to $[1,0]$, $[0,1]$, $[1,1]$ by a projective transformation

So any P_0, P_1, P_2, P_3 is projectively equivalent to $[1,0], [0,1], [1,1], [x_3, y_3]$ and now cross-ratio is

$$\frac{(1.1-1.0)(0.y_3-x_3.1)}{(1.y_3-0.x_3)(0.1-1.1)} = x_3/y_3$$

writing $[x_3, y_3] = [\lambda, 1]$ with $\lambda = x_3/y_3$, we have $P_0 P_1 P_2 P_3$ is proj. equivalent to $[1,0]$, $[0,1]$, $[1,1]$, $[\lambda,1]$ where $\lambda = (P_0P_1 : P_2P_3)$

so 2 quadruples with same cross-ratio are equivalent to the same point hence to each other.

(c). By composing with a suitable proj. transformation ϕ we have $\phi\circ\tau$ fixes $[1,0]$, $[0,1]$, $[1,1]$ (τ is 1-1)

But $\phi\circ\tau$ preserves crossratio, so the argument of (b) shows it must fix all points in $\mathbb{CP}^1$!

So $\phi\circ\tau = id$ and $\tau = \phi^{-1}$ is projective.

(d). swapping 0,1 indices in the formula in (a) inverts the expression, as does swapping 2,3 indices.

$0 \leftrightarrow 2$
$1 \leftrightarrow 3$ also leaves top, bottom unchanged

For last part, take $P_0 = [1,0]$, $P_1 = [0,1]$, $P_2 = [1,1]$, $P_3 = [\lambda, 1]$ so cross-ratio $= \lambda$.

Now $(P_0 P_2 : P_1 P_3) = ([1 0], [1 1] : [0,1], [\lambda, 1])$

$$= \frac{(1.1 - 0.0)(1.1 - \lambda.0)}{(1.1 - \lambda.0)(1.1 - 1.0)}$$

$$= 1 - \lambda$$

(e). First part of (d) shows $V_4 \leq S_4$ leaves cross-ratio invariant. So we get six values permuted by S_3

$$\lambda, \quad 1-\lambda, \quad 1/\lambda, \quad \frac{1}{1-\lambda}, \quad \frac{\lambda}{\lambda-1}, \quad \frac{\lambda-1}{\lambda}$$

(Recall $\lambda \neq 0,1$ as pts are distinct.)

So if we have < 6 distinct values then these are 1, 2 or 3

We get equality if

$\lambda = 1-\lambda \iff \lambda = 1/2$

$\lambda = 1/\lambda \iff \lambda = -1$ as $\lambda \neq 1$

$\lambda = \frac{1}{1-\lambda} \iff \lambda^2 - \lambda + 1 = 0$ no real root

$\lambda = \frac{\lambda}{\lambda - 1} \iff \lambda = 2$

$\lambda = \frac{\lambda - 1}{\lambda} \iff \lambda^2 - \lambda + 1 = 0$ no real root

we obtain $-1, \frac{1}{2}, 2$.

let $P_0 = [1 0]$
$P_1 = [x_1, 1]$
$P_2 = [x_2, 1]$
$P_3 = [x_3, 1]$

$$(P_0 P_1 : P_2 P_3) = \frac{(1.1 - 0.x_2)(x_1 - x_3)}{(1.1 - 0.x_3)(x_1 - x_2)} = \frac{x_1 - x_3}{x_1 - x_2}$$

This $= -1$ iff $x_1 - x_3 = x_2 - x_1$

$\iff x_1 = \frac{x_2 + x_3}{2}$

i.e. x_1 is midpoint of $\overrightarrow{x_2 x_3}$