H.1

(a) The criterion for a 4×4 matrix L to represent a Lorentz transformation is that $L^t g L = g$, where g is the matrix $\mathrm{diag}(1, -1, -1, -1)$. Equivalently, $L^{-1} = g L^t g$.

L is proper and orthochronous if $\det L = 1$ and $L^0_0 > 0$.

[4 marks]

This is standard course work.

(b) For the 'if' part, check that

$$\begin{pmatrix} \gamma & \gamma u/c \\ \gamma u/c & \gamma \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \gamma & \gamma u/c \\ \gamma u/c & \gamma \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and note the positivity of the 00 component and the determinant.

[4 marks]

For the 'only if', let the matrix be $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$.

The Lorentz conditions give $p^2 - r^2 = 1, pq = rs, q^2 - s^2 = -1$. The extra conditions give $p > 0, ps - qr = 1$.

Define $u = rc/p = qc/s$, noting that $|u| < c$. Then $\gamma = p$, using the fact that p is positive, and $r = \gamma u/c$.

The determinant condition also implies that $1 - qr/ps = 1/ps$, i.e. $1 - u^2/c^2 = 1/ps$, and hence s is positive.

Now similarly we have $\gamma = s$ and $q = \gamma u/c$.

[7 marks]

This argument has been studied in the class worksheets.

For the rapidity, define ϕ by $\gamma u/c = \sinh \phi$, then $\gamma = \cosh \phi, u/c = \tanh \phi$ and the matrix is $\begin{pmatrix} \cosh \phi & \sinh \phi \\ \sinh \phi & \cosh \phi \end{pmatrix}$, with an immediate additivity property.

[3 marks]

Also standard.

(c) The rapidity associated with two neighbouring walkways is $\tanh^{-1}(1/N)$. This is additive. Hence the rapidity associated with the motion of P_N relative to P_0 is

$N \tanh^{-1}(1/N)$. So the relative velocity is

$$\tanh(N \tanh^{-1}(1/N))c\,.$$

[4 marks for this or an equivalent statement]

Use first-year algebra of limits and continuity to determine
$\lim_{N\to\infty} N \tanh^{-1}(1/N) = \lim_{x\to 0} x^{-1} \tanh^{-1} x = 1$ by L'Hôpital,
hence $\tanh(N \tanh^{-1}(1/N))c \to (\tanh 1)c = \frac{e^2-1}{e^2+1}c$.

Or use a series: $\tanh^{-1}(x) = \frac{1}{2}\ln((1+x)/(1-x)) = x + x^3/3 + x^5/5 \ldots$.

[3 marks for taking the limit correctly]

This question is intended to be a little easier than last year's, which proved too demanding for a second-year course.