1(a)

$H = y'\frac{\partial K}{\partial y'} - K$ is indep of x so,

$$0 = \frac{dH}{dx} = y''\frac{\partial K}{\partial y'} + y'\frac{d}{dx}\left(\frac{\partial K}{\partial y'}\right) - \frac{\partial K}{\partial x} - \frac{\partial K}{\partial y}y' - \frac{\partial K}{\partial y'}y''$$

$\frac{\partial K}{\partial x} \to 0$ (K indep of x

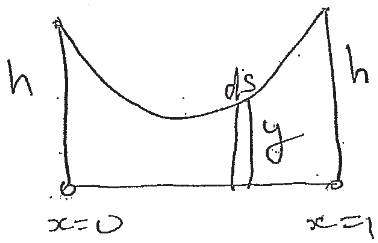
$$= y'\left[\frac{d}{dx}\left(\frac{\partial K}{\partial y'}\right) - \frac{\partial K}{\partial y}\right] \quad \text{"} 0$$

$= 0$

$\therefore$ H is const.

/3 Bookwork

(b) /18



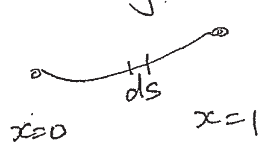
(i) $y(0) = y(1) = h$ /1

(ii) $PE = \int_0^1 \rho g y\, ds = \int_0^1 \rho g\sqrt{1+y'^2}\, y\, dx$ /2

since $ds = \sqrt{dx^2 + dy^2}$

$\therefore f = \rho g\sqrt{1+y'^2}$

Chain has length L

but  $L = \int_0^1 ds = \int_0^1 \sqrt{1+y'^2}\, dx$ /1

$p = \sqrt{1+y'^2}$

(iii) Here, extremal of

$$J(y) = \rho g \int_0^1 y\sqrt{1+y'^2}\,dx$$

is same as that of $\int_0^1 y\sqrt{1+y'^2}\,dx$

So w.l.o.g. take

$$f = y\sqrt{1+y'^2}$$

Now f & p indep of x,

so $y'\frac{\partial K}{\partial y'} - K$ is const.

i.e. $y'\frac{\partial}{\partial y'}\left[y\sqrt{1+y'^2} - \lambda\sqrt{1+y'^2}\right] - \sqrt{1+y'^2}\,(y-\lambda) = \text{const}$

$$\Rightarrow y'\left[(y-\lambda)\frac{y'}{\sqrt{1+y'^2}}\right] - \sqrt{1+y'^2}\,(y-\lambda) = \text{const}$$

$$\Rightarrow \frac{y-\lambda}{\sqrt{1+y'^2}}\left[y'^2 - (1+y'^2)\right] = \text{const}$$

Bookwork

$$\Rightarrow \frac{y-\lambda}{\sqrt{1+y'^2}} = \text{const}$$

$$\Rightarrow \frac{(y-\lambda)^2}{1+y'^2} = c_1^2 \text{ (const)}$$

4

Not standard

(iv) $c_1 = 0 \Rightarrow y = \text{const}$

But $y = \text{const} \Rightarrow I(y) = \int_0^1 \sqrt{1+0^2}\,dx = 1 < L$ ✗.

2

/12

(c)(i) $\frac{(y-\lambda)^2}{1+y'^2} = c_1^2$

$\Rightarrow \frac{(y-\lambda)^2 - c_1^2}{c_1^2} = y'^2$

$\Rightarrow \frac{dy}{dx} = \pm\sqrt{\frac{(y-\lambda)^2 - c_1^2}{c_1^2}}$

$\Rightarrow \frac{dy}{\sqrt{\left(\frac{y-\lambda}{c_1}\right)^2 - 1}} = \pm\, dx$

Set $u = \frac{y-\lambda}{c_1}$

$\therefore du = \frac{1}{c_1} dy$

using hint: $c_1 \cosh^{-1} u = \pm x + \hat{\gamma}$ ← const.

analysis standard but different to standard bookwork

$\therefore \cosh^{-1} u = \pm c_1^{-1} x + \gamma$ const

$\therefore u = \cosh(\pm c_1^{-1} x + \gamma)$

$\therefore y = \lambda + c_1 \cosh(\pm c_1^{-1} x + \gamma)$

/3

$y(0) = h \Rightarrow h - \lambda = c_1 \cosh \gamma$

$y(1) = h \Rightarrow h - \lambda = c_1 \cosh(\gamma \pm \frac{1}{c_1})$

$\Rightarrow \cosh \gamma = \cosh(\gamma \pm \frac{1}{c_1})$

$\Rightarrow \gamma = \gamma \pm \frac{1}{c_1} \Rightarrow c_1 = \infty \Rightarrow y = \infty$ X. finite chain length

or $\gamma = -\gamma \mp \frac{1}{c_1}$

$\Rightarrow \gamma = \mp \frac{1}{2c_1}$

$\therefore y = \lambda + c_1 \cosh\left(\pm \frac{x}{c_1} \mp \frac{1}{2c_1}\right)$

$= \lambda + c_1 \cosh\left(\frac{x}{c_1} - \frac{1}{2c_1}\right)$ Since $\cosh(\theta) = \cosh(-\theta)$

/3

ii) $L = \int_0^1 \sqrt{1+y'^2}\, dx = \int_0^1 \sqrt{1+\sinh^2\left(\frac{x}{c_1} - \frac{1}{2c_1}\right)}\, dx$

$= \pm \int_0^1 \left[\cosh\left(\frac{x}{c_1} - \frac{1}{2c_1}\right)\right] dx$

$= \pm c_1 \left[\sinh\left(\frac{x}{c_1} - \frac{1}{2c_1}\right)\right]_0^1$

$$\Rightarrow L = \pm c_1 \left[\sinh\left(\frac{1}{c_1} - \frac{1}{2c_1}\right) - \sinh\left[-\frac{1}{2c_1}\right] \right]$$

$$= \pm c_1 \left[2\sinh\frac{1}{2c_1} \right]$$

L must be +ve

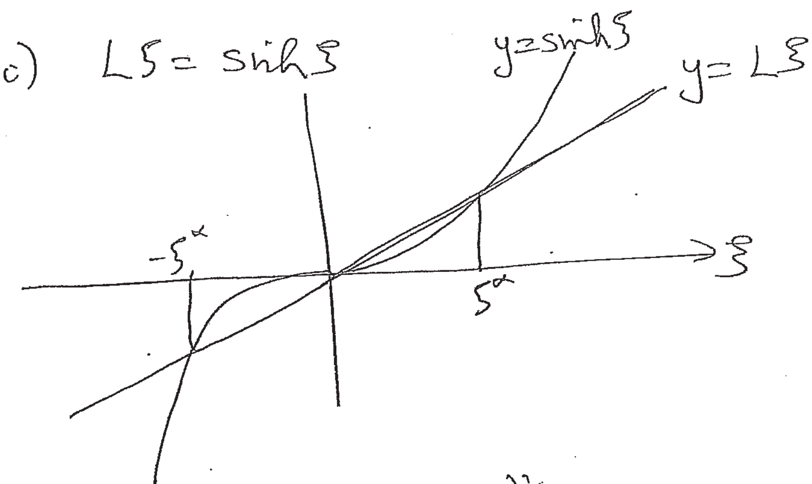
$\therefore$ w.l.o.g. take c_1 +ve

$$\therefore L = 2c_1 \sinh\frac{1}{2c_1}$$

$$\Rightarrow L\xi = \sinh\xi, \quad \xi = \frac{1}{2c_1}$$

(N.B. ξ could still be −ve & this works) ~~2~~

(iii) $L\xi = \sinh\xi$



Soln $\xi = 0 \Rightarrow (c = \infty)\ y = \infty$ ✗.

$\xi = \pm\xi^\alpha \Rightarrow c_1 = \pm c_1^\alpha$ & since c_1 occurs inside cosh, w.l.o.g. take c_1 +ve.
& $y'(0) = -\sinh\left(\frac{1}{c}\right) < 0$ for min. $\Rightarrow c > 0$ ~~2~~ 3

(iv) $y = \lambda + \frac{1}{2\xi^\alpha}\cosh(2\xi^\alpha x - \xi^\alpha)$

but $y(0) = h$ $\therefore$ $h = \lambda + \frac{1}{2\xi^\alpha}\cosh(-\xi^\alpha)$

$$\therefore \lambda = h - \frac{1}{2\xi^\alpha}\cosh(\xi^\alpha)$$

$$\therefore y = h + \frac{1}{2\xi^\alpha}\left[\cosh((2x-1)\xi^\alpha) - \cosh\xi^\alpha\right]$$ ~~3~~ 2