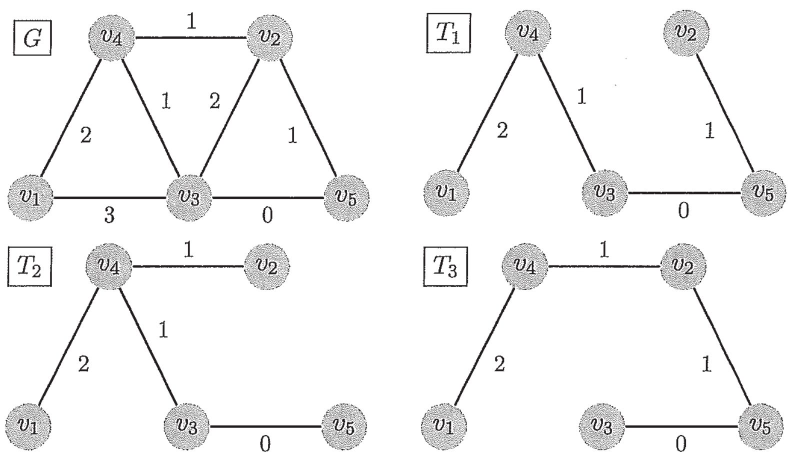
Graph Theory

(a) [5 marks] Let G be the graph with $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$ and $E(G) = \{v_1v_3, v_1v_4, v_2v_3, v_2v_4, v_2v_5, v_3v_4, v_3v_5\}$. Define a cost function $c : E(G) \to \mathbb{R}$ by $c(v_1v_3) = 3$, $c(v_1v_4) = 2$, $c(v_2v_3) = 2$, $c(v_2v_4) = 1$, $c(v_2v_5) = 1$, $c(v_3v_4) = 1$, $c(v_3v_5) = 0$. Draw all minimum cost spanning trees in G and prove that your list is complete.



3 [B]

As $|V(G)| = 5$, any spanning tree contains 4 edges. The minimum sum achieved by 4 edges of G is $0+1+1+1 = 3$; however, these edges do not form a tree. The next smallest sum is $0+1+1+2 = 4$. The is a unique way to choose the 0-edge and three ways to choose the two 1-edges. There is then a unique way to choose the 2-edge so that the resulting graph is a tree. Therefore T_1, T_2 and T_3 are the minimum cost spanning trees of G. *2 [S]*

Another valid proof that the list is complete is to consider all possible runnings of an algorithm for finding a minimum cost spanning tree and show that it must terminate with one of T_1, T_2 or T_3.

(b) [10 marks] Let G be a connected finite graph and define

$$\mathcal{F} = \{A \subseteq E(G) : (V(G), A) \textbf{ is a forest}\},$$
$$\mathcal{T} = \{A \subseteq E(G) : (V(G), A) \textbf{ is a tree}\}.$$

(i) Prove that if $A \subseteq B \in \mathcal{T}$ then $A \in \mathcal{F}$, and that for every $A \in \mathcal{F}$ there is B with $A \subseteq B \in \mathcal{T}$.

Suppose first that $A \subseteq B \in \mathcal{T}$. Then $(V(G), B)$ is a tree, so is acyclic. As $A \subseteq B$, clearly A is also acyclic, so by definition $(V(G), A)$ is a forest. *1 [B]*

Now suppose $A \in \mathcal{F}$. Consider B of maximum size subject to $A \subseteq B$ and $(V(G), B)$ being acyclic. Suppose for a contradiction that $(V(G), B)$ is not a tree. Then $(V(G), B)$ is not connected. Fix x and y in different components of $(V(G), B)$. As G is connected, there is a path P in G from x to y. As P does not remain within a single component of $(V(G), B)$, we can fix some edge e of P with endpoints in different components of $(V(G), B)$. Then $(V(G), B \cup \{e\})$ is acyclic, contradicting maximality of B, so $B \in \mathcal{T}$. *4 [B]*

(ii) Suppose $A \in \mathcal{T}$, $B \in \mathcal{F}$ and $e \in E(G)$ with $B \setminus A = \{e\}$. Prove that there is $C \in \mathcal{T}$ and $f \in A$ such that $B \subseteq C$ and $C = (A \setminus \{f\}) \cup \{e\}$.

Suppose that $e = xy$. As $(V(G), A)$ is connected, it contains a path P from x to y. Adding e to P completes a cycle. This cycle cannot be contained in B, as B is acyclic, so we can fix $f \in (P \cup \{e\}) \setminus B$. As $e \in B$, we have $f \in P \subseteq A$. Let $C = (A \setminus \{f\}) \cup \{e\}$. Then $B \subseteq C$. *3 [S]*

As $|C| = |A|$, to show that C is a tree it suffices to show that C is connected, i.e. that for any vertices s and t we can find a walk in C from s to t. To see this, consider any walk W in A from s to t, which exists because A is connected. By replacing any use of f in W by the path $(P \cup \{e\}) \setminus \{f\}$ we obtain a walk in C from s to t. *2 [B]*

(c) [10 marks] Let X be a finite set. Suppose that $\mathcal{F}$ and $\mathcal{T}$ are sets of subsets of X satisfying properties (i) and (ii) of (b). Let $c : X \to \mathbb{R}$ be a non-negative cost function. For $A \subseteq X$ write $c(A) = \sum_{x \in A} c(x)$. Consider the following algorithm:

1. **Let $A_0 = \emptyset$ and $i = 0$.**
2. **Let $Y_i = \{x \in X \setminus A_i : A_i \cup \{x\} \in \mathcal{F}\}$.**
3. **If $Y_i \neq \emptyset$ then choose $x_{i+1} \in Y_i$ such that $c(x_{i+1}) = \min_{x \in Y_i} c(x)$, let $A_{i+1} = A_i \cup \{x_{i+1}\}$, increase i by 1, and return to step 2.**
4. **If $Y_i = \emptyset$ then output $A = A_i$.**

Prove that the output A of the algorithm satisfies $c(A) = \min_{B \in \mathcal{T}} c(B)$.

We first claim that $A \in \mathcal{T}$. To see this we note by (b)(i) that $A_0 = \emptyset \in \mathcal{F}$, and if $A_i \in \mathcal{F}$ for some i with $A_i \neq A$ then $A_{i+1} \in \mathcal{F}$, so by induction $A \in \mathcal{F}$. Now by (b)(i) there is B with $A \subseteq B \in \mathcal{T}$. If we have $B \neq A = A_i$ then for any $e \in B \setminus A$ we have $e \in Y_i$, as $A_i \cup \{e\} \in \mathcal{F}$ by (b)(i). However, in this case the algorithm would not have terminated, so in fact $B = A \in \mathcal{T}$. *4 [S/N]*

Now let $m = \min_{B \in \mathcal{T}} c(B)$ and $\mathcal{M} = \{B \in \mathcal{T} : c(B) = m\}$. We prove by induction on i that there is B with $A_i \subseteq B \in \mathcal{M}$. Note that when applied to $A_i = A$ this will prove our required statement, as then $c(A) \leq c(B) = m$, so $c(A) = m$ by minimality of m, as $A \in \mathcal{T}$. *2 [S]*

For the base case of the induction we have $A_i = \emptyset$, so we can fix any $B \in \mathcal{M}$. For the induction step, suppose we have $A_i \subseteq B \in \mathcal{M}$ and $A_i \neq A$. If $A_{i+1} = A_i \cup \{x_{i+1}\} \subseteq B$ then then induction step is complete. Otherwise, we note that $B \in \mathcal{T}$, $A_{i+1} \in \mathcal{F}$ and $A_{i+1} \setminus B = \{x_{i+1}\}$, so by (b)(ii) there is $C \in \mathcal{T}$ and $f \in B$ with $A_{i+1} \subseteq C$ and $C = (B \setminus \{f\}) \cup \{x_{i+1}\}$. As $A_i \cup \{f\} \subseteq B$ we have $A_i \cup \{f\}$ by (b)(i), so $f \in Y_i$. Then $c(x_{i+1}) \leq c(f)$ by minimality in the algorithm, so $c(C) = c(B) - c(f) + c(x_{i+1}) \leq c(B) = m$, so $C \in \mathcal{M}$ by minimality of m. This completes the induction, and so proves the required statement. *4 [S/N]*