Solution:

[B] (a)(i) [2 marks] A finite group G is said to be solvable if there is a series of subgroups

$$\{e\} = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$$

such that the quotient groups G_i/G_{i-1} are abelian for $i = 1, \ldots, n$. (If the above is a composition series then the quotient groups will be cyclic and of prime order.)

[B/S] (ii) [4 marks] A_4 is solvable as

$$\{e\} \lhd V_4 \lhd A_4$$

which has abelian factors V_4 and C_3.

[B] (b) [6 marks in total] We'll show that the composition factors of G are those of N and G/N put together. To see this, let

$$\{e\} = \bar{K}_0 \lhd \bar{K}_1 \lhd \cdots \lhd \bar{K}_n = G/N$$

be a composition series for G/N with abelian factors, and let

$$\{e\} = N_0 \lhd N_1 \lhd \cdots \lhd N_k = N$$

be one for N. [2] The subgroups $\bar{K}_i \leqslant G/N$ correspond to subgroups $K_i \leqslant G$ containing N, and there are isomorphisms

$$K_i/K_{i+1} \cong (K_i/N)/(K_{i+1}/N) = \bar{K}_i/\bar{K}_{i+1}$$

by the Third Isomorphism Theorem. [3] Hence we get a composition series

$$\{e\} = N_0 \lhd N_1 \lhd \cdots \lhd N_k = N = K_0 \lhd K_1 \lhd \cdots \lhd K_n = G.$$

with abelian factors. [1]

[N] (c) (i) [4 marks] The first column of an invertible 2×2 with entries in $\mathbb{Z}_3$ can be any non-zero vector in $(\mathbb{Z}_3)^2$, and there are 8 such vectors. Such a column vector has 3 scalar multiples and the second column can be none of these, leaving 6 possibilities. Hence $|G| = 8 \times 6 = 48$.

[S/N] (c) (ii) [2 marks] Every element of G has determinant ± 1. The subgroup $N = \mathrm{SL}(2,3)$ of elements with determinant 1 is normal as N is the kernel of the the determinant homomorphism There are equal numbers with each determinant (e.g. multiplying the first row by -1 provides a bijection between these) and so $|N| = 24$.

[N] (c) (iii) [5 marks] Note that $\{\pm I_2\}$ is a normal subgroup of N. So $N/\{\pm I_2\}$ is a group of order 12. Consider coset representatives

$$\alpha = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

in $N/\{\pm I_2\}$. Note that

$$\begin{aligned}
\alpha^3 &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = I_2; \\
\beta^2 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -I_2; \\
(\alpha\beta)^3 &= \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} = I_2.
\end{aligned}$$

Hence we have a well-defined homomorphism

$$A_4 = \langle \alpha, \beta \,|\, \alpha^3 = \beta^2 = (\alpha\beta)^3 = 3 \rangle \to N/\{\pm I_2\}$$

induced by $\alpha \mapsto \alpha, \beta \mapsto \beta$.

As α has order 3 and β has order 2 in $N/\{\pm I_2\}$ then the image of this homomorphism has order 6 or 12 and so its kernel has order 1 or 2. But A_4 has no normal subgroup of order 2 and the homomorphism is an isomorphism.

[S/N] (c) (iv) [2 marks] As $\{\pm I_2\}$ is solvable and A_4 is solvable then N is solvable by (b). As $G/N \cong C_2$ and N are solvable then G itself is solvable (by (b) again).