Solution:

[B] (a)(i) [5 marks in total] The Laplace transform of $f_n(x)$ equals

$$\overline{f_n}(p) = \int_0^\infty x^n e^{-px}\,\mathrm{d}x.$$

and is defined for $p > 0$ (or $\mathrm{Re}\,p > 0$). [1 mark]

Note that $\overline{f_0}(p) = p^{-1}$ [1 mark] and by integration by parts we have [2 marks]

$$\overline{f_n}(p) = \left[-\frac{1}{p}x^n e^{-px}\right]_0^\infty + \frac{n}{p}\int_0^\infty x^{n-1}e^{-px}\,\mathrm{d}x = \frac{n}{p}\overline{f_{n-1}}(p).$$

Hence [1 mark]

$$\overline{f_n}(p) = \frac{n}{p} \times \frac{n-1}{p} \times \cdots \times \frac{1}{p} \times \overline{f_0}(p) = \frac{n!}{p^{n+1}}.$$

[B/S] (a)(ii) [3 marks] For $a > -1$, the Laplace transform of x^a equals

$$\begin{aligned}\int_0^\infty x^a e^{-px}\,\mathrm{d}x &= \int_0^\infty \left(\frac{u}{p}\right)^a e^{-u}\,\frac{\mathrm{d}u}{p} \qquad [u = px]\\ &= \frac{1}{p^{a+1}}\int_0^\infty u^a e^{-u}\,\mathrm{d}u\\ &= \frac{\Gamma(a+1)}{p^{a+1}}.\end{aligned}$$

[B] (b)(i) [2 marks] Note that e^x has Laplace transform

$$\int_0^\infty e^x e^{-px}\,\mathrm{d}x = \int_0^\infty e^{-(p-1)x}\,\mathrm{d}x = \frac{1}{p-1}$$

and so e^x is the required inverse Laplace transform.

[S/N] (b)(ii) [4 marks in total] The Laplace transform of x^{k-1} equals $\Gamma(k)/p^k$ [1 mark] and of $e^x f(x)$ equals $\overline{f}(p-1)$ [1 mark]. Hence the required inverse Laplace transform is

$$\frac{x^{k-1}e^x}{\Gamma(k)} \qquad \text{[2 marks]}.$$

[N] (b)(iii) [4 marks in total] From (a)(ii) we know that $x^{-1/2}$ has Laplace transform $\sqrt{\pi/p}$ [1 mark] Hence the inverse Laplace transform of $(p-1)^{-1}p^{-1/2}$ is the convolution of e^x and $x^{-1/2}/\sqrt{\pi}$ [1 mark for saying convolution]. This equals

$$\begin{aligned}&\int_0^x \frac{1}{\sqrt{\pi}\sqrt{t}}e^{(x-t)}\,\mathrm{d}t \qquad \text{[1 mark for this expression]}\\ =\ & \frac{e^x}{\sqrt{\pi}}\int_0^x \frac{1}{\sqrt{t}}e^{-t}\,\mathrm{d}t\\ =\ & \frac{e^x}{\sqrt{\pi}}\int_0^{\sqrt{x}} \frac{1}{u}e^{-u^2}\,2u\mathrm{d}u \qquad [t = u^2]\\ =\ & e^x\frac{2}{\sqrt{\pi}}\int_0^{\sqrt{x}} e^{-u^2}\,\mathrm{d}u\\ =\ & e^x\mathrm{erf}(\sqrt{x}). \qquad \text{[1 mark for correct answer]}\end{aligned}$$

[N] (c) [7 marks in total] We have

$$
\begin{aligned}
\overline{\mathrm{erf}}(p) &= \frac{2}{\sqrt{\pi}} \int_{x=0}^{\infty} \int_{t=0}^{x} e^{-t^2} e^{-px} \, \mathrm{d}t \, \mathrm{d}x && \text{[1 mark for this expression]} \\
&= \frac{2}{\sqrt{\pi}} \int_{t=0}^{\infty} \int_{x=t}^{\infty} e^{-t^2} e^{-px} \, \mathrm{d}x \, \mathrm{d}t && \text{[swapping limits – 2 marks]} \\
&= \frac{2}{\sqrt{\pi}} \int_{t=0}^{\infty} e^{-t^2} \left[\frac{e^{-px}}{-p} \right]_t^{\infty} \mathrm{d}t \\
&= \frac{2}{p\sqrt{\pi}} \int_{t=0}^{\infty} e^{-t^2} e^{-pt} \, \mathrm{d}t && \text{[1 mark for this expression]} \\
&= \frac{2e^{p^2/4}}{p\sqrt{\pi}} \int_{t=0}^{\infty} e^{-(t+p/2)^2} \, \mathrm{d}t && \text{[1 mark for completing square]} \\
&= \frac{e^{p^2/4}}{p} \frac{2}{\sqrt{\pi}} \int_{u=p/2}^{\infty} e^{-u^2} \, \mathrm{d}u \\
&= \frac{1}{p} e^{p^2/4} \left(\mathrm{erf}(\infty) - \mathrm{erf}(p/2) \right) && \text{[1 mark for this rearrangement]} \\
&= \frac{1}{p} \exp\left(\frac{p^2}{4} \right) \left(1 - \mathrm{erf}\left(\frac{p}{2} \right) \right) && \text{[1 mark for final answer].}
\end{aligned}
$$