ASO Introduction to Manifolds

[B] 1(a). [5 marks]

$f: \mathbb{R}^n \to \mathbb{R}^k$ is <u>differentiable</u> at x if there is a linear map $L_x : \mathbb{R}^n \to \mathbb{R}^k$ such that

$$f(x+h) = f(x) + L_x(h) + R_x(h)$$

where $\frac{\|R_x(h)\|}{\|h\|} \to 0$ as $h \to 0$.

f is differentiable on $\mathbb{R}^n$ if it is differentiable at each point x.

[B] (b). [5 marks]

<u>Inverse Function Theorem:</u>

Let $f: U \subset \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable, and suppose its derivative $df(x_0)$ at $x_0 \in U$ is invertible. Then f is a local diffeomorphism on some neighbourhood V of x_0 i.e. $f: V \to f(V)$ is a differentiable bijection with differentiable inverse.

(c). $GL(n,\mathbb{R}) = \det^{-1}(\mathbb{R}-\{0\})$

[5] which is open as det is a polynomial function so continuous.

[5 marks]

$$(I+h)^{-1} = I - h + o(h)$$

so $d(Inv)_I : h \mapsto -h$

now $(A+h)^{-1} = (A(I+A^{-1}h))^{-1}$: $A \in GL(n,\mathbb{R})$

$$= (I+A^{-1}h)^{-1}A^{-1}$$

$$= (I - A^{-1}h + o(h))A^{-1}$$

$$= A^{-1} - A^{-1}hA^{-1} + o(h)$$

so $d(Inv)_A : h \mapsto -A^{-1}hA^{-1}$

(d).

[N]

[5 marks]

$$\exp(h) = I + h + h^2/2! + \dots$$

$$= I + h + \sum_{n=2}^{\infty} h^n/n!$$

now $\left\| \sum_{n=2}^{\infty} h^n/n! \right\| \leq \sum_{n=2}^{\infty} \|h\|^n/n!$

$$= e^{\|h\|} - \|h\| - 1$$

Now $\lim\limits_{t\to 0} \frac{e^t - t - 1}{t} = 0$ (e.g. by L'Hôpital)

so $\exp(h) = \exp(0) + h + R(h)$: $\|R(h)\| / \|h\| \to 0$ as $h \to 0$

so $d(\exp)_0 : h \mapsto h$

i.e. $d(\exp)_0$ is the identity.

Inverse Function Thm $\Rightarrow$ exp gives a local diffeomorphism between a nbd. of 0 and a nbd. of I.

[N] (e).

[5] marks

if $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $B^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2+bc & b(a+d) \\ c(a+d) & d^2+bc \end{pmatrix}$

if this $= \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$

then offdiagonal terms $\Rightarrow$ $a+d \neq 0$
$c = 0$

so $B^2 = \begin{pmatrix} a^2 & b(a+d) \\ 0 & d^2 \end{pmatrix}$

and as $a,b,c,d \in \mathbb{R}$ we cannot have $d^2 = -1$. So $\nexists$ a square root of $\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$

(e) continued)

Now the hint tells us in particular that

$$\exp\left(\frac{X}{2}\right) \cdot \exp\left(\frac{X}{2}\right) = \exp(X)$$

so $\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$ is not of the form $\exp(X)$ for any $X \in M_{n\times n}(\mathbb{R})$.