1

A50 Projective Geometry

[B] (a). Let $B : V \times V \to \mathbb{F}$ be a bilinear form, $\not\equiv 0$. Suppose $\dim V = 3$

[3 marks]

Then the <u>conic</u> C in the plane $\mathbb{P}(V)$ associated to B is defined by:

$$\{ [u] : B(u,u) = 0 \}$$

The conic is <u>nonsingular</u> if B is nondegenerate. (i.e. $B(u,v) = 0 \ \forall v \Rightarrow u = 0$)

[5] (b). A line is given by $\{ [\lambda u + \mu v] : \lambda, \mu \in \mathbb{C} \}$ where u, v are linearly independent. The equation

[4 marks]

$B(\lambda u + \mu v, \lambda u + \mu v) = 0$ is now

(*) $$\lambda^2 B(u,u) + 2\lambda\mu B(u,v) + \mu^2 B(v,v) = 0$$

The coefficients cannot all be zero as then, in a basis $\{u, v, w\}$, $B = \begin{bmatrix} 0 & 0 & \cdot \\ 0 & 0 & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$

which means B is degenerate, contradicting nonsingularity of C. So the eqn. is a homogeneous quadratic with either 2 distinct roots $[\lambda, \mu]$ or 1 (double) root.

(c). As B is bilinear, $B(w,\cdot) \in V^*$

[B]

and $\phi : w \mapsto B(w,\cdot)$ is linear.

Non degeneracy $\iff$ ϕ is 1-1

[6] marks

and hence is an isomorphism, as $\dim V = \dim V^*$

$B(w,\cdot) \in U^0 \iff B(w,u) = 0 \quad \forall u \in U$
$\iff w \in U^\perp$

so $\dim U^\perp = \dim U^0 = \dim V - \dim U$, as required.

$\dim \langle u \rangle = 1$, $\dim V = 3$, so $\dim \langle u \rangle^\perp = 2$

and $p^\perp = \mathbb{P}(\langle u \rangle^\perp)$ is a projective line.

Clearly $U \subseteq U^{\perp\perp}$, and from above the dimensions are equal, so $U = U^{\perp\perp}$.

(d). Let $P = [u] \in C$. So $B(u,u) = 0$, and if we

[N]

consider as in (b) a line $\{[\lambda u + \mu v] : \lambda, \mu \in \mathbb{C}\}$, then eqn. $(*)$ becomes

$$2\lambda\mu \, B(u,v) + \mu^2 B(v,v) = 0$$

[4] marks

i.e. $\mu = 0$ (giving P), or

$$2\lambda B(u,v) + \mu B(v,v) = 0$$

The line is tangent $\iff$ the second root is also $\mu = 0$

$\iff B(u,v) = 0$

$\iff u, v \perp u$ as $B(u,u) = 0$

(to get uniqueness of tangent) $\iff$ line is $P^\perp$

(e). Suppose $P \notin C$

[N] [5 marks]

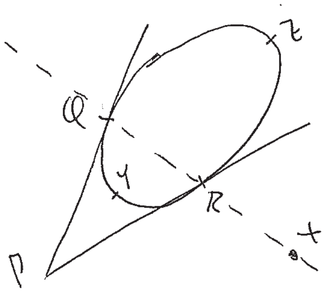
If $P^\perp$ meets C at a unique point Q then $P^\perp$ is the tangent to C at Q so, by (d), $P^\perp = Q^\perp$, so $P = Q$

Hence, by (b), $P^\perp$ meets C in 2 distinct points $Q = [v_1]$ and $R = [v_2]$.

Let $P = [u]$.

so $B(u, v_1) = B(v_1, v_1) = 0$, so the line through P, Q is $Q^\perp$, which by (d) is the tangent to C at Q. Similarly the line through P, R is $R^\perp$ which is the tangent to C at R.

[N] (f). [3 marks]



Let $X \in P^\perp$, $\neq Q, R$.

From (e), the tangents to C meeting at X meet C at the points $C \cap X^\perp$: call these pts Y, Z

But $X \in P^\perp \Rightarrow P \in X^\perp$

so P, Y, Z are collinear