Special Relativity

Solution:

1. (a) $V = (V^0, \mathbf{v})$ is future pointing time-like if $V^0 > 0$ and $(V^0)^2 - \mathbf{v}\cdot\mathbf{v} > 0$, i.e., $V^0 > |\mathbf{v}|$. [2]

We can orient the axes so $\mathbf{v} = (v, 0, 0)$, $v \geqslant 0$ and by assumption $V^0 > v$. Then with $u/c = -v/V^0$, the standard Lorentz transformation is the identity in the (y, z) variables and in the (ct, x) variables becomes

$$L = \gamma(u)\begin{pmatrix} 1 & \frac{u}{c} \\ \frac{u}{c} & 1 \end{pmatrix} = \frac{1}{\sqrt{g(V,V)}}\begin{pmatrix} V^0 & -v \\ -v & V^0 \end{pmatrix}$$

so $LV = \sqrt{g(V,V)}(1,0)$. [$\cancel{2}$] 3

Squaring the desired inequality and expanding out the left hand side, we find that we have to see that

$$g(V,V) + 2g(V,W) + g(W,W) \geqslant g(V,V) + 2\sqrt{g(V,V)g(W,W)} + g(W,W)\,,$$

so it suffices to prove $g(V,W) \geqslant \sqrt{g(V,V)g(W,W)}$ but in a frame in which $V = (V^0, 0)$ the left hand side is V^0W^0 whereas the right is $V^0\sqrt{(W^0)^2 - |\mathbf{w}|^2}$ so the inequality is clear with equality iff $\mathbf{w} = 0$ i.e., if V and W are proportional. [3]

In the twin scenario given, the lengths of V, W and $V+W$ have the interpretation of proper time along the worldline along $V+W$, V and W respectively. The inequality then implies that the observer along $V+W$ experiences a greater elapse in proper time than that along V and then W, i.e., the twin paradox. [$\cancel{2}$] 3

All bookwork and lecture notes at this point.

(b) For a massive particle $P = mV$ where m is the rest mass and V the 4-velocity of the particle defined by the condition that it is tangent to the worldline of the particle with $g(V,V) = c^2$. The energy according to U is $E = g(P,U)/c$. For a massless particle we take P to be tangent to the worldline with scale fixed by the energy relation given by $E = \hbar\nu$ for a photon of frequency ν. (*Still just definitions.*)[3]

By conservation of momentum $P = P_1 + P_2$ is the 4-momentum of the coalesced particle and the inequality of the previous part immediately gives the desired inequality. Equality happens only if P, P_1 and P_2 are proportional, so the additional mass corresponds to the assimilation of the kinetic energies of particles one and two into the rest mass of P. [3]

Let $P_1 = m_1(c, 0)$ and $P_2 = m_2\gamma(u)(c, u)$ then the mass M is given by $M^2c^2 = g(P_1 + P_2, P_1 + P_2) = m_1^2c^2 + m_2^2c^2 + 2m_1m_2c^2\gamma$ and the speed v/c is given as the ratio of the space-like to the timelike part of $P_1 + P_2$ as

$$\frac{v}{c} = \frac{m_2\gamma(u)u}{m_1c + m_2\gamma(u)c}\,.$$

[$\cancel{6}$] 3

(c) If now we have a photon with 4-momentum P decomposing into an electron/positron pair with 4-momenta P_1 and P_2 then we have $P = P_1 + P_2$, but then $g(P_1 + P_2, P_1 + P_2) \geqslant 2m^2c^2$ from above but this contradicts P being null. [1]

If now the photon hits a nucleon with 4-momentum P_N, then we will have

$$P + P_N = P_N' + P_1 + P_2\,.$$

Extending the inequality of the first part to three particles in the obvious way, we have

$$\sqrt{g(P+P_N)} \geqslant \sqrt{g(P'_N, P'_N)} + \sqrt{g(P_1, P_1)} + \sqrt{g(P_2, P_2)} = (M+2m)c\,,$$

with equality if P_1, P_2 and P'_N are proportional (i.e., all have the same velocity). [3]

With $P = \frac{E}{c}(1,1)$ and $P_N = M(c,0)$ we obtain $\sqrt{2EM + M^2c^2}$ for the left hand side. The threshold energy is therefore where we have equality and squaring this gives

$$2EM + M^2c^2 = (M+2m)^2c^2\,,$$

so

$$E = \frac{2(M+m)mc^2}{M}\,.$$

When M is very large compared to m, E becomes $2mc^2$ giving the rest energy of the particles $2mc^2$ so it becomes 100%. [~~6~~] 3