1(a) E-L eqns: $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0.$ 1

[B]

$F = F(x, y, y') = F(y, y')$ in this case:

$$\frac{dF}{dx} = \underbrace{\cancel{\frac{\partial F}{\partial x}}}_{0} + y'\frac{\partial F}{\partial y} + y''\frac{\partial F}{\partial y'}$$

$$= y'\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + y''\frac{\partial F}{\partial y'}$$

(from E-L)

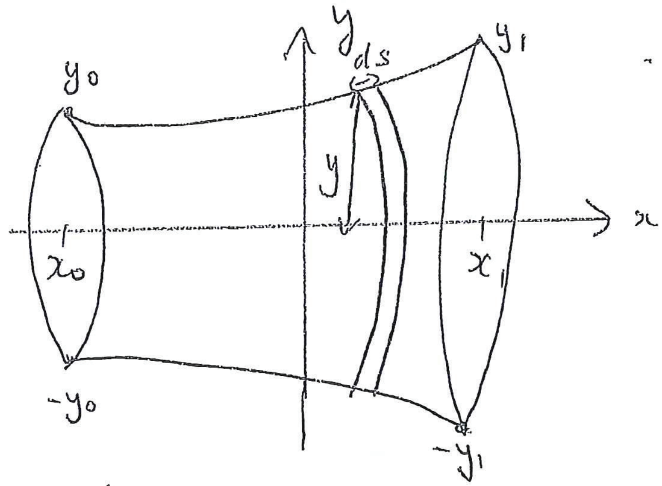
$$= \frac{d}{dx}\left(y'\frac{\partial F}{\partial y'}\right)$$

Hence $\frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y'}\right) = 0$

$$\therefore \quad F - y'\frac{\partial F}{\partial y'} = K \text{ const.}$$ 3

(b) (ii)

[B]



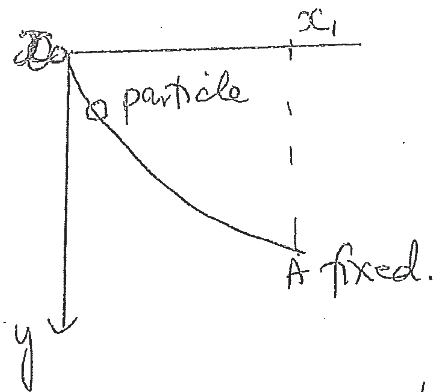
y fixed at the fixed pts x_0, x_1

$$dA = 2\pi y\, ds = 2\pi y \sqrt{1+y'^2}\, dx$$

$$\therefore A = 2\pi \int_{x_0}^{x_1} y \sqrt{1+y'^2}\, dx$$

3

(b) (ii)



y fixed at x_0 (w.l.o.g. set this to 0)

y fixed at x_1

$$\text{Energy} = \tfrac{1}{2}mv^2 - mgy = \text{const} \quad (\text{initial cond}^{\text{ns}})$$

$$\therefore v = \pm\sqrt{2gy} = \sqrt{2gy} \quad \text{for the case above (w.l.o.g.)}$$

$$\therefore dt = \frac{ds}{\sqrt{2gy}} \qquad \left(v = \frac{ds}{dt}\right)$$

$$\therefore T = \int_0^T dt = \int_{x_0}^{x_1} \frac{1}{\sqrt{2gy}} \left[\sqrt{1+y'^2}\right] dx.$$

$ds = \sqrt{dx^2 + dy^2}$

s = arc length

$\frac{ds}{dx} \Big/ dy$

$ds = dx\sqrt{1+y'^2}$

$$T = \frac{1}{\sqrt{2g}} \int_{x_0}^{x_1} y^{-1/2}\left(1+y'^2\right)^{1/2} dx$$

/3

(c) $F = y^n(1+y'^2)^{1/2}$

[N - they have not seen this way of solving the eqns]

$$\frac{\partial F}{\partial y'} = \frac{y^n y'}{(1+y'^2)^{1/2}}$$

$$F - y'\frac{\partial F}{\partial y'} = K \Rightarrow y^n\left[1+y'^2\right]^{1/2}\left[1 - \frac{y'^2}{1+y'^2}\right] = K$$

$$\Rightarrow y^n\left[1+y'^2\right]^{-1/2} = K$$

$\tan\psi = y'$

$\therefore 1+y'^2 = 1+\tan^2\psi = \sec^2\psi$

$\therefore \underline{y^n\cos\psi = \pm K = \text{const.}}$

$\cancel{2}$ 3

$$\therefore ny^{n-1}\frac{dy}{d\psi}\cos\psi - y^n\sin\psi = 0$$

ie $n\frac{dy}{d\psi} = y\tan\psi$

$$\tan\psi = \frac{dy}{dx} = \frac{dy}{d\psi}\cdot\frac{d\psi}{dx}$$

$$\therefore n\frac{dy}{d\psi} = y\frac{dy}{d\psi}\left(\frac{dx}{d\psi}\right)^{-1}$$

$$\therefore \underline{y = n\frac{dx}{d\psi}}$$

$\cancel{3}$ 2

(5)

(i) $n=-\frac{1}{2} \Rightarrow y^{-1/2} = \frac{\hat{K}}{\cos\psi}$ $\quad \hat{K}$ const

$\therefore y = \pm\hat{K}^{-1}\cos^2\psi$ $\quad C$ const

$y = C(1+\cos 2\psi)$ $\quad$ absorb $\pm$ into C $\quad \cos^2\psi = \frac{1}{2}(1+\cos 2\psi)$

$\therefore -\frac{1}{2}\frac{dx}{d\psi} = C(1+\cos 2\psi)$

$\therefore \frac{dx}{d\psi} = -2C(1+\cos 2\psi)$

$\therefore x = C_1 - C(2\psi + \sin 2\psi)$ $\quad$ where C_1 is a const.

2

(ii) $n=1 \Rightarrow y = K\sec\psi$

$\frac{dx}{d\psi} = K\sec\psi$

$\therefore x = K_1 \pm K\ln|\sec\psi+\tan\psi|$ $\quad$ where K_1 is a const.

$\sec\psi+\tan\psi > 0 \Rightarrow \frac{x-K_1}{K} = \ln(\sec\psi+\tan\psi)$

$\therefore e^{\frac{x-K_1}{K}} = \sec\psi+\tan\psi$

Also: $-\left(\frac{x-K_1}{K}\right) = -\ln(\sec\psi+\tan\psi)$

$e^{-\left(\frac{x-K_1}{K}\right)} = \sec\psi - \tan\psi$

$\therefore e^{\frac{x-K_1}{K}} + e^{-\left(\frac{x-K_1}{K}\right)} = 2\sec\psi$

$\therefore y = K\cosh\left(\frac{x-K_1}{K}\right)$

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$y(-1) = y(1) = \alpha$

$\therefore \quad \alpha = K\cosh\left(\frac{1-K_1}{K}\right) = K\cosh\left(\frac{-1-K_1}{K}\right)$

$\Rightarrow \quad 1-K_1 = \pm(-1-K_1)$

$\Rightarrow K_1 = 0$

$\therefore \quad \underline{\frac{\alpha}{K} = \cosh\left(\frac{1}{K}\right)}$

/2

$$A = 2\pi \int_{-1}^{1} K\cosh\left(\frac{x}{K}\right)\sqrt{1+\sinh^2\frac{x}{K}}\,dx$$

$$= 2\pi \int_{-1}^{1} K\cosh^2\frac{x}{K}\,dx$$

$$= 2\pi \frac{K \times 2}{4}\int_{0}^{1}\left(e^{\frac{2x}{K}} + e^{-\frac{2x}{K}} + 2\right)dx$$

(or use double-angle formula) $A = \pi K \int_0^1 (\cosh 2x + 2)\,dx$

$$= \frac{\pi K}{2}\left[\frac{K}{2}e^{\frac{2x}{K}} - \frac{K}{2}e^{-\frac{2x}{K}} + 2x\right]_0^1$$

$$= \frac{\pi K^2}{2}\left[\sinh\frac{2x}{K} + \frac{2x}{K}\right]_0^1$$

$$= \frac{\pi K^2}{2}\left[\sinh\frac{2}{K} + \frac{2}{K}\right]$$

$$\underline{A = \pi K^2\left[\sinh\frac{2}{K} + \frac{2}{K}\right]}$$

/2