Graph Theory Solutions

Throughout this question, we let G be a connected graph and $c(e) > 0$ for each $e \in E(G)$. For $S \subseteq E(G)$ we let $c(S) = \sum_{e \in S} c(e)$.

(a) [5 marks] (i) Define an Euler tour in G and a postman walk in G (ii) State the Chinese Postman Problem (CPP)

(i) An Euler tour is a closed walk using every edge exactly once. A postman walk is a closed walk using every edge at least once. *3 [B]*

(ii) The CPP is to find a postman walk W minimising $c(W)$. *2 [B]*

(Abbreviations: B 'bookwork', S 'similar', N 'new'.)

(b) [8 marks] State and prove Euler's Theorem on Euler tours.

Euler's Theorem: a graph has an Euler tour if and only if it is connected and has all degrees even. *2 [B]*

Fleury's Algorithm: Start at any vertex and follow any walk, erasing each edge after it is used (erased edges cannot be used again), erasing each vertex when it becomes isolated, subject to not making the current graph disconnected. *2 [B]*

Proof of Theorem: We show that Fleury's Algorithm produces an Euler tour. To see this, suppose for a contradiction that it fails. This can only happen if the walk arrives at some vertex v such that however we continue the walk the graph will become disconnected. There must be at least two edges in the current graph that contain v, as if there are none then by connectivity the walk has used all edges, so we are done, or if there is exactly one then we delete v after using it, so the graph remains connected, and the walk can continue. As there are at least two edges containing v, we can choose one of them vw, such that the component C of $G - vw$ which contains w does not contain the first vertex u of the walk. But then w is the only vertex of odd degree in C, which is impossible ($\sum_v d(v) = 2e$ is always even). *4 [B]*

(c) [12 marks] Let X be the set of vertices with odd degree in G. Define a weight function w on pairs in X, where each $w(xy)$ is the minimum value of $c(P)$ among all xy-paths P in G. Let M be a perfect matching on X minimising $w(M)$.

(i) Describe how the data in M can be used to solve the CPP. [You are not required to prove optimality of the solution.]

(ii) The Efficient Postman Problem (EPP) is to find a closed walk W in G using every edge at most once that maximises $c(W)$. Suppose $X = \{x, y\}$ and $G \setminus P$ is connected for any xy-path P in G. Describe how a solution to the EPP can be obtained from the data in M, and prove that your solution is optimal.
(iii) Give an example of a graph G with $|X| = 2$ where the solution of the EPP cannot be obtained from the data in M as in (ii).

(i) For each $xy \in M$ let P_{xy} be an xy-path that minimises $c(P)$. Let G^* be obtained from G by copying all edges of P_{xy} for all $xy \in M$. Then G^* is connected and has all degrees even, so has an Euler tour W, which we can interpret as a postman walk in G. *2 [B]*

(ii) Let P_{xy} be as in (i). Let $G_1 = G - P_{xy}$ be obtained from G by deleting all edges of P_{xy}. Then G_1 has all degrees even and is connected (by assumption on G), so has an Euler tour W_1. We will prove that W_1 is an optimal solution to the EPP. To see this, consider any closed walk W in G using every edge at most once. Let $H = G - W$. Note that $X = \{x, y\}$ is also the set of vertices of odd degree in H. Now x and y cannot be in different components of H (the sum of degrees in any component is even), so there is an xy-path P in H. Note that $c(H) \geq c(P) \geq c(P_{xy})$ by definition of P_{xy}. Now $c(W_1) = c(G) - c(P_{xy}) \geq c(G) - c(H) = c(W)$, so W_1 is optimal. *6 [S]*

(iii) For an example where the method in (ii) doesn't work, consider the graph G on 6 vertices, labelled $\{1, \ldots, 6\}$, and 8 edges, consisting of triangles on 123 and 456, and also 14 and 25. Define $c(14) = c(25) = 1$ and $c(e) = 2$ for any edge in the triangles. Then $M = \{14, 25\}$, $P_{14} = 14$, $P_{25} = 25$, but deleting 14 and 25 leaves a disconnected graph. The solution to the EPP is obtained by deleting 12 and 45. *4 [N]*