ASO Group Theory 2019

[Let G be a finite group of order $|G|$ and let p and q be distinct primes.

(1) State Sylow's Theorem for G.]

Writing $|G| = p^\alpha m$ for $\alpha \geq 0$ and $(m,p)=1$,

[4] then (1) there exists a subgroup of order p^α,

(2) all such subgroups are conjugate

(3) if n_p denotes the number of such subgroups

bookwork then $n_p \equiv 1 \pmod{p}$ and $n_p \mid m$.

[Explain what it means for G to be soluable.]

G is soluable if there exists a subnormal sequence

[2] $G = G_n \triangleright G_{n-1} \triangleright \cdots \triangleright G_0 = \{e\}$

bookwork with subquotients abelian.

(Note: Equivalently, G has a composition series with composition factors abelian, or equivalently all composition factors are cyclic.)

[Henceforth you may assume that all definitions of soluable are equivalent.]

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[Prove that if $|G| = p^2$ then G is abelian.]

G acts on itself $(X = G)$ by conjugation.
Then $\text{Fix}_G X = C(G)$, the centre of G.
As $e \in C(G)$, and

[4]

$$|C(G)| = |\text{Fix}_G X| \equiv |X| = p^2 \equiv 0 \pmod p$$

Seen similar

$C(G)$ is non-trivial and $|C(G)| = p$ or p^2.

If $a \neq e \in C(G)$ with order $o(a) = p^2$ then $C(G) \simeq C_{p^2}$ and is abelian.

If there is no such a then there exists $a \neq e \in C(G)$ with $o(a) = p$.

Let $b \neq e \in G \setminus \langle a \rangle$. Then $|\langle b \rangle| = p$.

[3] As $\langle a \rangle \subseteq C(G)$ it is normal and hence

Seen similar

$$G = \langle a \rangle \langle b \rangle \quad \text{as } \langle a \rangle \cap \langle b \rangle = \{e\}$$

Also, b (and its multiples) commute with elements in $\langle a \rangle$ (as $\langle a \rangle \subseteq C(G)$).

Hence

$$G \simeq \langle a \rangle \times \langle b \rangle \simeq C_p \times C_p \qquad \square$$

Hint: You may use without proof a formula relating the size of a set X on which G acts and the size of its fixed point set when $|G| = p^\alpha$

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[(2) Prove that if $|G| = pq$ or $|G| = pq^2$ then G is soluable.]

Assume $|G| = pq$ and w.l.o.g $p < q$.
Then, by Sylow's Theorem, for some $k \geq 0$
$$n_q = kq+1 \mid p$$
and hence $n_q = 1$.
[4] This proves that the q-Sylow subgroup S_q is normal.
Hence

Seen similar
$$G = G_2 \triangleright S_q = G_1 \triangleright \{e\} = G_0$$
with $G_2/G_1 \cong C_p$ and $G_1/G_0 \cong C_q$. □

Assume $|G| = pq^2$.
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If $p < q$ then as above, $n_q = kq+1 \mid p$ and hence $n_q = 1$.
Then
$$G = G_2 \triangleright S_q = G_1 \triangleright \{e\} = G_0$$
[2]
and $|G_2/G_1| = p^2$ and hence G_2/G_1 is C_{p^2} or $C_p \times C_p$
So once again, the sub quotients are abelian
(or equivalently have cyclic composition factors)

If $p > q$ then by the Sylow theorem for some k
[2] $$n_p = kp+1 \mid q^2$$
Hence, $n_p = 1$ or $n_p = kp+1 = q^2$.
[1] If $n_p = 1$ then $G \triangleright S_p \triangleright \{e\}$ gives a subnormal sequence with abelian sub quotients
If $n_p = q^2$ then there must be
$(p-1)q^2$ many elements of order p, leaving
[3] only q elements to form a unique q-Sylow subgroup (and hence normal)
So $G \triangleright S_q \triangleright \{e\}$ works. □

[/12]