For Tutors Only - Not For Distribution

Integral Transforms Solution

(a)

(i) [$2 = 1 + 1$; B.] A test function is a function which has derivatives of all orders and vanishes outside a finite interval of $\mathbb{R}$. A distribution is a continuous linear map from the space of test functions to $\mathbb{R}$.

(ii) [$4 = 1 + 1 + 2$ (of which 1 for noting $x\phi$ is a test function); B.]

$$\langle \delta, \phi \rangle = \phi(0); \quad \langle \delta', \phi \rangle = -\phi'(0), \quad \langle x\delta, \phi \rangle = \langle \delta, x\phi \rangle = 0 \quad \text{(as } x\phi \text{ is a test function)}.$$

(iii) [$3 = 1 + 2$; B/S.]

$$\langle x\delta', \phi \rangle = \langle \delta', x\phi \rangle = -(x\phi)'|_{x=0} = -\phi(0)$$

(or could do Leibniz on $x\delta$);

$$\langle x^2\delta'', \phi \rangle = -\langle \delta', (x^2\phi)' \rangle = \langle \delta, (x^2\phi)'' \rangle = (x^2\phi)''|_{x=0} = 2\phi(0).$$

(b)

[This is all S with a touch of N. The example is new but the method will have been covered in lectures. The residue part needs accurate calculation; the contour choice tests understanding. $8 = 2$ for the jump at $x = 0$ + 2 for the transform + 1 for the inversion contour + 3 for the answer. If students solve by writing down exponentials and saying that y has a jump of -1 at $x = 0$ they get nothing (not following instructions). Students who invert by finding the FT of $\mathrm{e}^{-|x|}$ and saying that multiplying by $\mathrm{i}s$ is FT of derivative get full credit if correct.]

(i) [N; 2.] The most singular term on the left is $-y''$ and on the right we have δ' so, integrating once, y' has a $-\delta$ singularity at $x = 0$ and, integrating again, y has a jump of -1.
(ii) [S/N; $6 = 2 + 1 + 3$.] Taking the FT in x, noting that the FT of y' is $\mathrm{i}s\hat{y}$, we get

$$s^2\hat{y} + \hat{y} = \mathrm{i}s, \quad \text{so} \quad \hat{y} = \frac{\mathrm{i}s}{s^2 + 1}.$$

(FT of δ' is $\mathrm{i}s\hat{\delta}$, or via the action of δ' on $\mathrm{e}^{-\mathrm{i}xs}$.) Hence

$$y(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{i}s\mathrm{e}^{\mathrm{i}xs}}{s^2 + 1} \mathrm{d}s.$$

Using the hint, first take $x > 0$ so that we close the contour with a large semicircle in the UHP to get exponential decay from $\mathrm{e}^{\mathrm{i}xs}$ (Jordan's Lemma needed for the estimation but this is explicitly not asked for). The integrand has poles at $s = \pm\mathrm{i}$. The residue at $s = \mathrm{i}$ (in the contour) is

$$\left. \frac{\mathrm{i}s\mathrm{e}^{\mathrm{i}sx}}{s + \mathrm{i}} \right|_{s=\mathrm{i}} = \frac{\mathrm{i}^2\mathrm{e}^{-x}}{2\mathrm{i}} = \mathrm{i}\frac{1}{2}\mathrm{e}^{-x},$$

so for $x > 0$,

$$y(x) = 2\pi\mathrm{i} \times \frac{1}{2\pi} \times \mathrm{i}\frac{1}{2}\mathrm{e}^{-x} = -\frac{1}{2}\mathrm{e}^{-x}.$$

Hence

$$y(x) = \begin{cases} -\frac{1}{2}\mathrm{e}^{-|x|} & x > 0 \\ +\frac{1}{2}\mathrm{e}^{-|x|} & x < 0 \end{cases}$$

which indeed has a jump of -1 at $x = 0$.

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(c)

(i) [B, 2 = 1 + 1.]

$$[f * g](x) = \int_0^x f(t)g(x-t)\,\mathrm{d}t, \quad \overline{f * g} = \overline{f}\,\overline{g}.$$

(ii) [Calculation is a straightforward but unfamiliar S/N. 6 = 2 for finding ODE for $\overline{f}$ + 2 for solving + 2 for inversion.]

Taking the LT of the equation,

$$-\frac{\mathrm{d}}{\mathrm{d}p}(p\overline{f} - f_0) = \overline{f}^2,$$

where $f_0 = f(0)$, from which

$$p\frac{\mathrm{d}\overline{f}}{\mathrm{d}p} = -\overline{f} - \overline{f}^2,$$

so that

$$-\frac{\mathrm{d}p}{p} = \frac{\mathrm{d}\overline{f}}{\overline{f} + \overline{f}^2} = \left(\frac{1}{\overline{f}} - \frac{1}{\overline{f}+1}\right)\mathrm{d}\overline{f}.$$

Hence

$$-\log(p/p_0) = \log\overline{f} - \log(\overline{f}+1),$$

for arbitrary p_0, which gives

$$\frac{\overline{f}+1}{\overline{f}} = \frac{p}{p_0}$$

and then

$$\overline{f} = \frac{p_0}{p - p_0}.$$

Thus, $f(x) = p_0\mathrm{e}^{p_0 x}$ (and so $p_0 = f_0$). Check:

$$f * f = \int_0^x p_0\mathrm{e}^{p_0 t} \times p_0\mathrm{e}^{p_0(x-t)}\mathrm{d}t = p_0^2\mathrm{e}^{p_0 x}\int_0^x \mathrm{d}t = p_0^2 x\mathrm{e}^{p_0 x} = xf'.$$