Introduction to manifolds: solution

1. (a) A function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is *differentiable* if for every $x \in \mathbb{R}^n$, there exists a linear map $df(x)\colon \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{h\to 0} \frac{f(x+h) - f(x) - df(x)(h)}{|h|} = 0$$

We call $df(x)$ the *derivative* of f at x. [2]

The matrix form of the linear map $df(x)$ is given by the Jacobian matrix

$$Df(x) = \begin{pmatrix} \partial_1 f_1(x) & \dots & \partial_n f_1(x) \\ \vdots & \ddots & \vdots \\ \partial_1 f_m(x) & \dots & \partial_n f_m(x) \end{pmatrix}$$

of partial derivatives of f. [1]

Suppose that the function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ has continuous partial derivatives. Then f is differentiable. [1]

(b) The set $f^{-1}(0) \subset \mathbb{R}^n$ is a submanifold of $\mathbb{R}^n$, if for every $x \in f^{-1}(0)$, there exists an open neighbourhood U of x in $\mathbb{R}^n$ such that $\operatorname{rank} df(y) = m$ for all $y \in U$. The dimension of $f^{-1}(0)$ is $n - m$. [3]

(c) Let $f(x, y) = xy - 1$. The partial derivatives of f are clearly continuous, so $f\colon \mathbb{R}^2 \to \mathbb{R}$ is differentiable, with $df(x, y) = (y, x)$. [2]

The latter has rank 1 away from the origin, so everywhere along the hyperbola $f^{-1}(0)$. Thus $f^{-1}(0) \subset \mathbb{R}^2$ is a submanifold. [2]

(d) (i) Standard check. [1]

(ii) Let $\operatorname{Skew}(2n)$ be the linear space of $(2n) \times (2n)$ skew-symmetric matrices. Consider the function

$$f\colon M_{2n}(\mathbb{R}) \to \operatorname{Skew}(2n)$$

given by $f(X) = X^t \Omega X - \Omega$. A standard check shows that $f(X)$ is indeed skew-symmetric. [4]

At a point $X \in \operatorname{Sp}(n)$, the differential of f is given by

$$df(X)(H) = X^t \Omega H + H^t \Omega X.$$

[2]

Given a skew-symmetric matrix $Z \in \operatorname{Skew}(2n)$, set $H = -\frac{1}{2} X \Omega Z$. Then

$$\begin{aligned} df(X)(H) &= -\frac{1}{2} X^t \Omega X \Omega Z - \frac{1}{2} Z^t \Omega^t X^t \Omega X = -\frac{1}{2}\Omega^2 Z - \frac{1}{2} Z^t \Omega^t \Omega \\ &= \frac{1}{2} Z - \frac{1}{2} Z^t = Z \end{aligned}$$

since Z is skew-symmetric. So $df(X)$ is surjective. So indeed $\operatorname{Sp}(2n) = f^{-1}(0)$ is a submanifold of M_{2n}. [4]

(iii) We have $\dim M_{2n} = (2n)^2 = 4n^2$ and $\dim \operatorname{Skew}(2n) = 2n(2n-1)/2$ so

$$\dim \operatorname{Sp}(2n) = n(2n+1).$$

[3]

Commentary: (a) and (b) are bookwork and (c) is a small variant of a standard example. (d) is unseen, though it follows closely the treatment of O(n) *given in lectures.*