Modelling in Mathematical Biology

1. Consider the SIR model that describes an infection. The equations describing the time evolution of a population in the susceptible (S), infected (I), and recovered (R) compartments are

$$\begin{aligned}\frac{\mathrm{d}S}{\mathrm{d}t} &= \Lambda - \beta S I \frac{1}{N} - \mu S,\\ \frac{\mathrm{d}I}{\mathrm{d}t} &= \beta S I \frac{1}{N} - (\mu+\gamma) I,\\ \frac{\mathrm{d}R}{\mathrm{d}t} &= \gamma I - \mu R, \end{aligned} \tag{1}$$

where $N(t) = S(t) + I(t) + R(t)$ is the total population at time t and Λ, β, μ, γ are constant positive parameters.

(a) [3 marks] Give a biological interpretation of each of the parameters in the model. What are the dimensions of the parameters?

Solution

- Λ – is the constant population growth. Dimension is population/time
  β is the rate at which a susceptible becomes infected. Dimension is 1/time.
  μ is the death rate. Dimension is 1/time.
  γ is the rate at which infected recover. Dimension is 1/time.

[S – This is a standard application of material seen in lectures.]

(b) [6 marks] Find the equation for $\frac{\mathrm{d}N}{\mathrm{d}t}$ and solve. Show that $N(t) \to \frac{\Lambda}{\mu}$ as $t \to \infty$.

Solution Substitute the system of equations into: $\frac{\mathrm{d}N}{\mathrm{d}t} = \frac{\mathrm{d}S}{\mathrm{d}t} + \frac{\mathrm{d}I}{\mathrm{d}t} + \frac{\mathrm{d}R}{\mathrm{d}t}$

$$\begin{aligned}\frac{\mathrm{d}N}{\mathrm{d}t} &= \frac{\mathrm{d}S}{\mathrm{d}t} + \frac{\mathrm{d}I}{\mathrm{d}t} + \frac{\mathrm{d}R}{\mathrm{d}t},\\ &= \Lambda - \beta S I \frac{1}{N} - \mu S + \beta S I \frac{1}{N} - (\mu+\gamma) I + \gamma I - \mu R,\\ &= \Lambda - \mu(S+I+R),\\ &= \Lambda - \mu N\end{aligned}$$

Solve the ODE:

$$\begin{aligned}\frac{\mathrm{d}N}{\mathrm{d}t} &= \Lambda - \mu N,\\ \int \frac{dN}{\Lambda - \mu N} &= \int dt. \quad \text{Let } dy = \mu dN \Rightarrow dN = -dy/\mu,\\ \frac{-1}{\mu}\int \frac{dy}{y} &= \int dt,\\ \ln(y) &= \mu(c+t),\\ N &= \frac{\Lambda - e^{-\mu(c+t)}}{\mu}\end{aligned}$$

As $t \to \infty$ we can substitute into $N(t)$. Then e^{-t} as $t \to \infty$ will approach 0.
$\therefore N(t) = \frac{\Lambda}{\mu}$ when $t \to \infty$.

[S – A standard application of the material seen in lectures. Requires interpretation]

(c) [3 marks] Now let $N = \frac{\Lambda}{\mu}$ in (1) to form the limiting system. Explain why it is enough to only consider the first two equations of (1) when studying the dynamics of the limiting system.

Solution We know

$$\begin{aligned} \frac{\mathrm{d}N}{\mathrm{d}t} &= \frac{\mathrm{d}S}{\mathrm{d}t} + \frac{\mathrm{d}I}{\mathrm{d}t} + \frac{\mathrm{d}R}{\mathrm{d}t}. \quad \text{Since N is constant,} \\ 0 &= \frac{\mathrm{d}S}{\mathrm{d}t} + \frac{\mathrm{d}I}{\mathrm{d}t} + \frac{\mathrm{d}R}{\mathrm{d}t}, \\ \frac{\mathrm{d}R}{\mathrm{d}t} &= -\frac{\mathrm{d}S}{\mathrm{d}t} + -\frac{\mathrm{d}I}{\mathrm{d}t} \end{aligned}$$

Since N is constant and $\frac{\mathrm{d}R}{\mathrm{d}t}$ doesn't have any affect on $\frac{\mathrm{d}S}{\mathrm{d}t}$ or $\frac{\mathrm{d}I}{\mathrm{d}t}$, then $R(t)$ can be written in terms of $S(t)$ and $I(t)$.

[S – A standard application of the material seen in lectures.]

(d) [3 marks] Let $\hat{S} = \frac{S\mu}{\Lambda}$, $\hat{I} = \frac{I\mu}{\Lambda}$, $\tau = t\mu$, $\theta = \beta/\mu$ and $\xi = \gamma/\mu$. Show that the limiting system has non-dimensional form:

$$\begin{aligned} \frac{\mathrm{d}\hat{S}}{\mathrm{d}\tau} &= 1 - \theta\hat{S}\hat{I} - \hat{S}, \\ \frac{\mathrm{d}\hat{I}}{\mathrm{d}\tau} &= -(1+\xi)\hat{I} + \theta\hat{S}\hat{I} \end{aligned} \tag{2}$$

Solution Substitute in the new non-dimensional quantities given above and cancel out.

[S – A standard application of the material seen in lectures.]

(e) [10 marks] Determine the steady states of (2) and their linear stability, analytically or graphically. An epidemic occurs if the number of individuals infected increases, ie $\frac{\mathrm{d}\hat{I}}{\mathrm{d}\tau} > 0$. Suppose we start from the disease-free steady-state where everyone is susceptible. Calculate the basic reproductive rate R_0. Can there be an epidemic?

Solution

The steady states are $(1, 0)$ and the second $(\frac{1+\xi}{\theta}, \frac{1}{1+\xi} - \frac{1}{\theta})$ is biologically plausible (non-negative) only when $\theta \geqslant 1 + \xi$.

The Jacobian matrix is

$$\mathcal{J} = \begin{pmatrix} -\theta\hat{I} - 1 & -\theta\hat{S} \\ \theta\hat{I} & -(1+\xi) + \theta\hat{S} \end{pmatrix}.$$

At the steady state $(1, 0)$, we have

$$\mathcal{J} = \begin{pmatrix} -1 & -\theta \\ 0 & -(1+\xi) + \theta \end{pmatrix} \qquad \Longrightarrow \qquad \lambda = -1, -(1+\xi) + \theta.$$

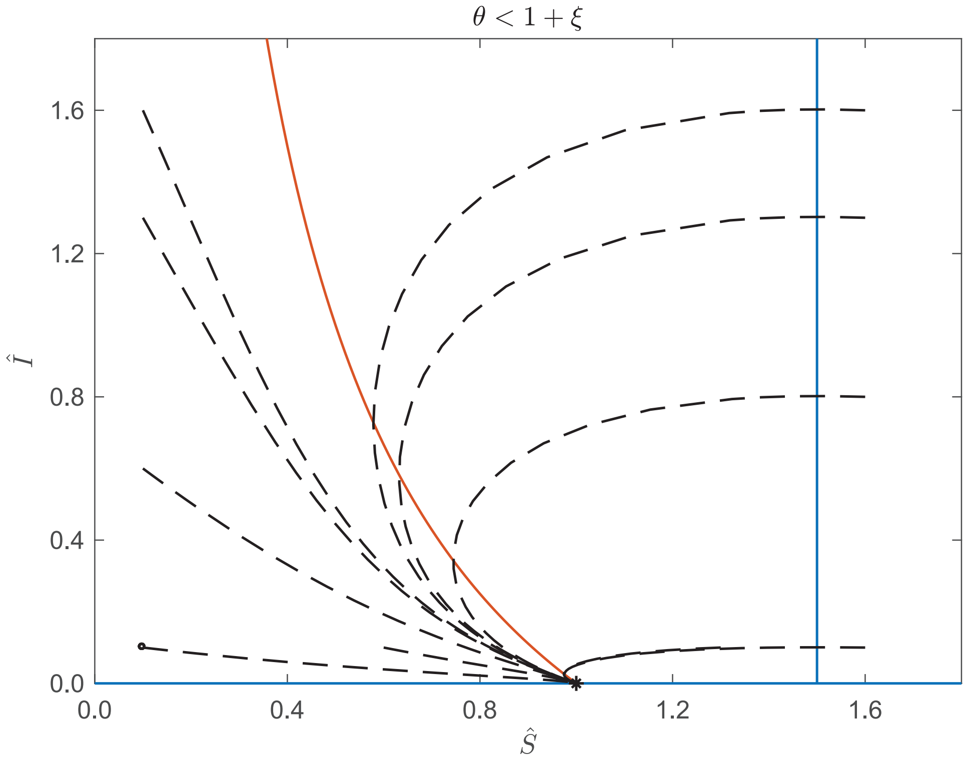
Hence $(1, 0)$ is a stable node when $\theta < 1 + \xi$.

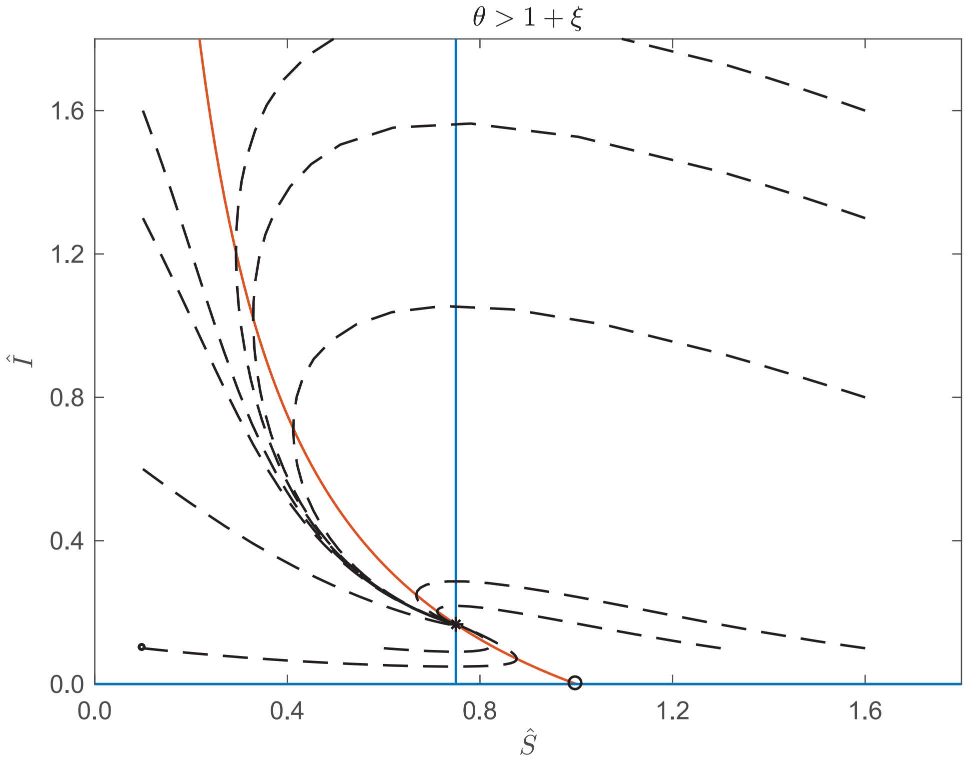
At the steady state $\frac{1+\xi}{\theta}, \frac{1}{1+\xi} - \frac{1}{\theta}$, we have

$$\mathcal{J} = \begin{pmatrix} -\frac{\theta-1-\xi}{1+\xi} - 1 & -(1+\xi) \\ \frac{\theta-1-\xi}{1+\xi} & 0 \end{pmatrix}$$

We can determine whether all the eigenvalues are negative using determinant and trace. The $det(\mathcal{J}) > 0$ when $\theta > 1 + \xi$, and the $tr(\mathcal{J}) < 0$. Hence $\frac{1+\xi}{\theta}, \frac{1}{1+\xi} - \frac{1}{\theta}$ is stable when $\theta > 1 + \xi$. Recall the parameter region that this steady-state is biologically plausible is when $\theta \geqslant 1 + \xi$.

Note that parameter values that admit a positive steady-state for the second steady-state are precisely the conditions when the disease-free equilibrium is unstable.





Let's start with our disease-free steady-state: $(1,0)$. We would like to determine the basic reproductive rate, which is the average number of secondary infections produced by one primary infection in a wholly susceptible population. We know if $R_0 > 1$ an epidemic occurs. If we take $\hat{S} = 1$ and substitute it into $\frac{\mathrm{d}\hat{I}}{\mathrm{d}\tau}$ to determine when the derivative is positive, we find:

$$\begin{aligned}
\frac{\mathrm{d}\hat{I}}{\mathrm{d}\tau} &= -(1+\xi)\hat{I} + \theta\hat{S}\hat{I}, \\
0 &< -(1+\xi)\hat{I} + \theta\hat{S}\hat{I}, \\
-\theta\hat{I} &< -(1+\xi)\hat{I}, \\
\frac{\theta}{(1+\xi)} &> \hat{I}/\hat{I}, \\
R_0 &> 1,
\end{aligned}$$

Therefore $R_0 = \frac{\theta}{(1+\xi)}$. We see in our analysis above that there is an epidemic if $\theta > 1+\xi$.

[S/N – Demanding good command of concepts. Requires determining (conditions) for real negative values) or plotting nullclines with different constraints on parameters. Manipulations more difficult than in lecture.]

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