2019 ASO Projective Geometry: solution

1. (a) (i) *Projective space* $\mathbb{P}(V)$ is the set of lines through the origin in a finite-dimensional vector space V over a field F.
(**Or**: $\mathbb{P}(V)$ is the set of nonzero vectors in V modulo the equivalence relation $\mathbf{v} \sim \lambda\mathbf{v}$ for $\lambda \in F^*$.)
A projective linear subspace S of $\mathbb{P}(V)$ is the set of lines in contained in a linear subspace $U \subset V$.
(**Or**: A projective linear subspace S of $\mathbb{P}(V)$ consists of equivalence classes represented by nonzero vectors contained in a linear subspace $U \subset V$.)
The dimension of S is $\dim S = \dim U - 1$ for such an S. *Lines* and *planes* are one, respectively two-dimensional projective linear subspaces of $\mathbb{P}(V)$. [3]

(ii) Given two projective linear subspaces L, M of $\mathbb{P}(V)$, let U, W be the corresponding linear subspaces of V. Then $\langle L, M\rangle$ is defined to be the projective linear subspace corresponding to the vector subspace $U + W$ of V. Geometrically, this is the union of lines PQ with $P \in L$, $Q \in M$. [2]
The formula is

$$\dim\langle L, M\rangle = \dim L + \dim M - \dim(L \cap M)$$

with the convention that if $L \cap M = \emptyset$ then $\dim(L \cap M) = -1$. To prove this, recall from Linear Algebra that for subspaces U, W of V as before, we have

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W).$$

Now simply subtract one from each term on each side to get the required equality, noting that $L \cap M = \emptyset$ if and only if $U \cap W = \{0\}$. [2]

(b) If L, M are two lines in $\mathbb{P}(V)$, then using the above formula, we obtain that

$$\dim\langle L, M\rangle = 2 - \dim(L \cap M).$$

Also $L \cap M \subset L$ so its dimension is at most one. So there are three cases:

1. $\dim(L \cap M) = 1$ so $L \cap M = L = M$ so the two lines coincide, and $\dim\langle L, M\rangle = 1$.
2. $\dim(L \cap M) = 0$ so $L \cap M = P$ is the unique intersection point of the two lines, and $\dim\langle L, M\rangle = 2$ the plane they span.
3. $\dim(L \cap M) = -1$ so $L \cap M = \emptyset$, the lines are disjoint, and $\dim\langle L, M\rangle = 3$.

[4]

(c) Let $S = \langle L, M\rangle$ and $T = \langle L, N\rangle$; by the discussion in (b), these are planes. Then by the dimension of intersection fornula,

$$\dim\langle S, T\rangle = 4 - \dim(S \cap T).$$

On the other hand, $L \subset S \cap T \subset S$ so either $S \cap T = S = T$ or $S \cap T = L$. [2]
In the first case, all of L, M, N are contained in the plane $S = T$ so we are done. In the second case, $\dim\langle S, T\rangle = 3$. Notice that $\langle S, T\rangle = \langle S, N\rangle$. Using the dimension of intersection formula again,

$$3 = \dim\langle S, N\rangle = 3 - \dim(S \cap N).$$

Thus $\dim S \cap N = 0$ so $S \cap N$ is a point Q. This is the only intersection point of N with the plane spanned by L and M so this point Q must also be on L and M. Thus Q is a common intersection point of all the three lines. [4]

(d) (i) Let $S = \langle P, L\rangle$ and $T = \langle P, M\rangle$. Since P is not contained in either of the lines, the dimension of intersection formula (or common sense) shows these are planes. Also they must be different planes, since otherwise L, M would be coplanar and hence should meet. Thus $\dim(S \cap T) < 2$. On the other hand,

$$\dim\langle S, T\rangle = 4 - \dim(S \cap T).$$

The left hand side must be at most $3 = \dim \mathbb{P}(V)$, so $\dim(S \cap T) > 0$. Thus $S \cap T$ is a line N which goes through P. It is also coplanar with L, N so it must intersect them.

(**Or:** By the discussion in (b), $\langle L, M\rangle$ is three-dimensional since L, M do not intersect, hence it must be the whole of $\mathbb{P}(V)$. Since this is the locus covered by lines through a point each of L and M, there must exist a line N incident with L, M and through P.)

[4]

(ii) Use coordinates $[x_0, x_1, x_2, x_3]$ on $\mathbb{P}^3$. The plane S above, containing the points $P_1 = [1, 0, 0, 0]$, $P_2 = [0, 1, 0, 0]$ and $P = [1, 1, 1, 1]$, has equation $x_2 - x_3 = 0$. Similarly, the plane T containing the points $P_3 = [0, 0, 0, 1]$, $P_4 = [0, 0, 1, 0]$ and $P = [1, 1, 1, 1]$, has equation $x_0 - x_1 = 0$. The intersection N of these two planes satisfies both these equations. We simply need to choose another point satisfying both equations, for example $Q = [1, 1, 0, 0]$. [4]

Commentary: (a) is bookwork. (b) is very similar to a discussion in lectures. (c) and (d) are unseen in this form, though similar in theme to problems on the problem sheets.