SOLUTIONS

The letters B, S and N written after the marks given below represent: bookwork material explicitly seen before, similar to matherial seen before and new rider respectively.

a. The criterion for a 4×4 matrix L to represent a Lorentz transformation is that L has real entries and satisfies

$$L^T g\, L = g \qquad g = \mathrm{diag}(+1, -1, -1, -1)\,,$$

where T denotes transposition *[2 marks,B]*. Lorentz transformations are called proper if they satisfy the further condition $\det(L) = 1$ *[1 mark,B]*. Lorentz transformations are called orthochronous if they satisfy the further condition ${L^0}_0 > 0$ *[1 mark,B]*.

Concerning the group structure of Lorentz transformations:

(i) *[2 marks,B]* To show that we have a group we have to check two basic properties: (A) closure and (B) existence of the inverse element. These are verified as follows

(A) Let L_1 and L_2 satisfy the condition $L_i^T g\, L_i = g$, with $i = 1, 2$ and let $L = L_1 L_2$ (where matrix multiplication is understood). A simple chain of identities

$$L^T g\, L = (L_1 L_2)^T g\, (L_1 L_2) = L_2^T\, L_1^T g\, L_1\, L_2 = L_2^T g\, L_2 = g\,,$$

shows that L satisfies the Lorentz condition.

(B) Existence of an inverse follows from the fact that $L^T g\, L = g$ implies that $(\det L)^2 = 1$ and in particular $\det L \neq 0$,

(ii) *[2 marks,S]*

(A) Closure of proper Lorentz transformations follows from the basic property of determinants $\det(L_1 L_2) = \det(L_1)\det(L_2)$.

(B) From the same property of the determinant it follows that $\det(L) = 1 \Rightarrow \det(L^{-1}) = 1$.

b.

(i) The rest mass of a massive particle is the ratio between an applied force and its acceleration in the ICS where the particle is instantaneously at rest. The four-momentum of a massive particle of rest mass m and velocity four vector U is the four-vector $P = m\,U$. Recall that $g(U, U) = c^2$ so that $g(P, P) = m^2\, c^2$ *[2 marks, B]*. The four-momentum of a massless particle satisfies the condition $g(P, P) = 0$, so its general expression is given by $P = (\frac{1}{c}E, \frac{1}{c}E\,\hat{\mathbf{n}})$ where $\hat{\mathbf{n}} \cdot \hat{\mathbf{n}} = 1$ *[1 mark, B]*.

(ii) *[4 marks, S]* Let us denote the incoming four-momenta as

$$p_1 = (\tfrac{1}{c}E, \vec{\mathbf{p}}) \qquad p_2 = (c\,M, \vec{\mathbf{0}}) \qquad \text{with } \tfrac{1}{c^2}E^2 - \vec{\mathbf{p}}\cdot\vec{\mathbf{p}} = M^2\,c^2\,,$$

and the outgoing momenta as p_3, p_4, p_π. Les us define and compute the quantity

$$s = g(p_1+p_2, p_1+p_2) = (\tfrac{1}{c}E + c\,M)^2 - \vec{\mathbf{p}}\cdot\vec{\mathbf{p}} = 2E\,M + 2c^2\,M^2\,.$$

By momentum conservation s is also equal to $g(p_3+p_4+p_\pi, p_3+p_4+p_\pi)$. Since s is Lorenz invariant it can be computed in any frame. Let us choose the frame in which $p_3+p_4+p_\pi = (\sqrt{s}, \vec{\mathbf{0}})$. In this frame $p_3 = (\frac{1}{c}E_3^*, \vec{\mathbf{p}}_3^*)$, $p_4 = (\frac{1}{c}E_4^*, \vec{\mathbf{p}}_4^*)$, $p_\pi = (\frac{1}{c}E_\pi^*, \vec{\mathbf{p}}_\pi^*)$ with

$$E_3^* \geqslant M\,c^2\,, \qquad E_4^* \geqslant M\,c^2\,, \qquad E_\pi^* \geqslant m\,c^2\,.$$

We conclude that

$$s = \tfrac{1}{c^2}\,(E_3^* + E_4^* + E_\pi^*)^2 \geqslant (M+M+m)^2\,c^2\,.$$

Recalling the expression of s in terms of E and M we thus have

$$2E\,M + 2c^2\,M^2 \geqslant (2M+m)^2\,c^2 \qquad \Rightarrow \qquad E \geqslant \frac{1}{2M}\left((2M+m)^2 - 2\,M^2\right)\,c^2\,.$$

The minimal energy so that the pion is produced is the one that saturates the inequality.

c.

(i) *[2 marks, S]* Lorentz transformations are defined by the condition $g(X_1, X_2) = g(LX_1, LX_2)$ for any vector X_1, X_2. The important observation to be made is that

$$2\,g(X_1, X_2) = x_1^+\,x_2^- + x_1^-\,x_2^+\,.$$

From this expression it is easy to see that the component of the Lorentz group connected to the identity corresponds to the transformations

$$(x^+, x^-) \mapsto (\lambda\,x^+, \lambda^{-1}x^-)\,, \qquad \lambda > 0\,,$$

and that the remaining group element are obtained by composing such transformations with

$$(x^+, x^-) \mapsto (x^-, x^+) \qquad (x^+, x^-) \mapsto (-x^+, -x^-)\,.$$

(ii) The parameter s measures proper time if

$$g(U,U) = c^2 \qquad (U^1, U^2) = (\tfrac{dc\,t(s)}{ds}, \tfrac{dx(s)}{ds})\,.$$

[1 mark, B] This condition can be rewritten in terms of $(x^+(s), x^-(s))$ as

$$\dot{x}^+\dot{x}^- = c^2\,, \qquad \Rightarrow \qquad \dot{x}^- = \frac{c^2}{\dot{x}^+}\,.$$

where we set $\frac{dx^+}{ds} = \dot{x}^+, \frac{dx^-}{ds} = \dot{x}^-$ *[1 mark,S]*.

(iii) *[2 marks, S]* The acceleration two-vector is defined as $A^a = \frac{dU^a}{ds}$. In terms of $(x^+(s), x^-(s))$ this reads

$$g(A,A) = \ddot{x}^+ \ddot{x}^- = \ddot{x}^+ \frac{d}{ds}\left(\frac{c^2}{\dot{x}^+}\right) = -c^2 \left(\frac{\ddot{x}^+}{\dot{x}^+}\right)^2$$

(iv) The general solution of the differential equation $g(A,A) = -\alpha^2$ is given by

$$\dot{x}^+ = c\,\lambda\, e^{+\sigma\,\frac{\alpha}{c}s}\,, \qquad \dot{x}^- = c\,\lambda^{-1}\, e^{-\sigma\,\frac{\alpha}{c}s}\,,$$

where $\sigma^2 = 1$ *[2 marks, N]*. The parameter λ and the sign σ are interpreted as Lorentz transformations acting on the space of solution. This is a symmetry of the problem so it must be present in the space of solutions as well *[2 marks, N]*.