EXTENSIONS OF FUNCTIONS

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ABSTRACT. The aim of the course is to present several results on extensions of functions. Among the most important are Kirszbraun's and Whitney's theorems. They provide powerful technical tools in many problems of analysis. One way to view these theorems is that they show that there exists an interpolation of data with certain properties. In this context they are useful in computer science, e.g. in clustering of data (see e.g. [26, 23]) and in dimension reduction (see e.g. [15]).

1. Syllabus

Lecture 1. McShane's theorem [25], Kirszbraun's theorem [18, 31, 35], Kneser-Poulsen conjecture [19, 29, 16].

Lecture 2. Whitney's covering and associated partition of unity, Whitney's extension theorem [37, 12, 33].

Lecture 3. Whitney's theorem – minimal Lipschitz extensions [22].

Lecture 4. Ball's extension theorem, Markov type and cotype [6].

2. Required mathematical background

Markov chains, Hilbert spaces, Banach spaces, metric spaces, Zorn lemma

3. Reading list

The reading list consists of all the papers cited above, lecture notes [27], and parts of books [36, 8].

4. Final assesment

Final assesment would comprise giving a talk on a topic related to the content of the course. Suggested topics include:

(1) Brehm's theorem [10],
(2) continuity of Kirszbraun's extension theorem [20],
(3) Kirszbraun's theorem for Alexandrov spaces [21, 1],
(4) two-dimensional Kneser-Poulsen conjecture [9],
(5) origami [11],
(6) absolutely minimising Lipschitz extensions and infinity Laplacian [17, 32, 34, 2, 3, 5, 4],
(7) Fenchel duality and Fitzpatrich functions [30, 7],
(8) sharp form of Whitney's extension theorem [13],
(9) Whitney's extension theorem for C^m [14],
(10) Markov type and cotype calculation [27, 6, 28],

(11) extending Lipschitz functions via random metric partitions [24, 27].

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