Twistor Charges for the S-algebra

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July 10, 2025

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Motivation

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- The algebra in question is the S-algebra between soft operators

$$[J^a[k,l,m], J^b[k',l',m']] = f_c^{ab} J[k+k', l+l', m+m']$$

with $k, l \in \mathbb{Z}_0^+, m \in \mathbb{Z}$. Seen from collinear limits of gluon scattering amplitudes (Strominger'21, Guevara, Himwich, Pate, Strominger'21).

- An amplitude is mapped to a correlation function

$$\mathcal{A}_n(p_1, \cdots, p_n) \underset{\text{Mellin}}{\overset{\text{Fourier}}{\longleftrightarrow}} \langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle$$

where $\mathcal{O}$ is an operator in the dual theory.

- The S-algebra is derived from the OPE of soft operators coming from collinear limits

$$\mathcal{A}_n(p_1, \cdots, p_n) \xrightarrow{p_1 || p_2} \text{Split}(p_1, p_2, p_1+p_2)\mathcal{A}_{n-1}(p_1+p_2, p_3, \cdots, p_n)$$

which, after the integral transform, gives the OPE of the soft operators.

Motivation

- Since then, the S-algebra has been studied in various ways. From a phase-space perspective (Freidel, Pranzetti, Raclariu '23 ,Cresto, Freidel '25), one constructs charges

$$Q_\alpha = -\frac{2}{g_{\mathrm{YM}}^2}\int_{\mathscr{I}} \mathrm{Tr}\left(\bar{F}(\partial_z \alpha_0 + [A, \alpha_0])\right)$$

  that generate variations on the radiative phase space.
- Under an appropriate symplectic structure

$$\Omega = \frac{2}{g_{\mathrm{YM}}^2}\int_{\mathscr{I}} \mathrm{Tr}\left(\delta\bar{F}\delta A\right) \qquad \{Q_\alpha, Q_{\alpha'}\} = Q_{[[\alpha,\alpha']]}$$

  the charges manifest the S-algebra.
- Computations become complicated due to the field dependence of the symmetry parameter α.

Motivation

- From a more top-down perspective, consider the holographic dictionary on twistor space (Gaiotto, Costello '18, Costello,Paquette '22, Costello, Paquette, Sharma '23), which is a 6d theory coupled to a 2d world volume theory

$$Z = \int DaDbD\psi \, e^{-S_{6d}[a,b]-S_{2d}[\psi]}$$

- To couple to the 6d theory, there must be currents in the 2d theory that couple to the bulk fields. For the path integral to be gauge invariant the currents must satisfy the OPE

$$J^a[k,l](z)J^b[k',l'](0) \sim \frac{f^{ab}_c}{z} J^c[k+k',l+l'](0)$$
$$J^a[k,l](z)\tilde{J}^b[k',l'](0) \sim \frac{f^{ab}_c}{z} \tilde{J}^c[k+k',l+l'](0)$$
$$\tilde{J}^a[k,l](z)\tilde{J}^b[k',l'](0) \sim 0$$

- This is the statement of Koszul duality. (Paquette, Williams '21)

Outline

Self-Dual Yang-Mills

Self-Dual Yang-Mills

- Self-dual Yang-Mills (SDYM) is an integrable subsector of full Yang-Mills, which sets the anti-self-dual (ASD) part of the field strength to zero.
- Decompose the field strength as

$$F_{\mu\nu} = F_{\alpha\dot{\alpha}\beta\dot{\beta}} = F_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}} + \tilde{F}_{\dot{\alpha}\dot{\beta}}\epsilon_{\alpha\beta}$$

where the ASD part is set to zero

$$F_{\alpha\beta} = \partial_{(\alpha}^{\ \dot{\alpha}} A_{\beta)\dot{\alpha}} + \frac{1}{2}[A_{(\alpha}^{\ \dot{\alpha}}, A_{\beta)\dot{\alpha}}] = 0$$

- The classical integrability of the theory is made manifest by its Lax pair

$$L_{\dot{\alpha}} = q^{\alpha} D_{\alpha\dot{\alpha}} = q^{\alpha}(\partial_{\alpha\dot{\alpha}} + A_{\alpha\dot{\alpha}})$$

where $q^{\alpha} = (1, q)$ is the spectral parameter.
- In terms of the Lax pair, the SDYM equations are given by

$$F_{\alpha\beta} = 0 \Leftrightarrow [L_{\dot{\alpha}}, L_{\dot{\beta}}] = 0 \quad \forall q$$

Chalmers-Siegel action

- We will also be interested in ASD perturbations $B_{\alpha\beta}$ satisfying

$$D^{\alpha\dot{\alpha}}B_{\alpha\beta} = \partial^{\alpha\dot{\alpha}}B_{\alpha\beta} + [A^{\alpha\dot{\alpha}}, B_{\alpha\beta}] = 0$$

- SDYM with an ASD perturbation is described by the Chalmers-Siegel action (Chalmers, Siegel 1996)

$$S_{CS}[B, A] = \int_{\mathcal{M}} d^4x \mathrm{Tr}\left(B_{\alpha\beta}F^{\alpha\beta}\right)$$

- To recover the full Yang-Mills action, add just one term

$$S_{YM} = S_{CS} + \frac{\epsilon}{2}\int_{\mathcal{M}} d^4x B_{\alpha\beta}B^{\alpha\beta}$$

which gives the equations of motion

$$D^{\alpha\dot{\alpha}}B_{\alpha\beta} = 0 \qquad F_{\alpha\beta} = \epsilon B_{\alpha\beta}$$

Recursion Relations

- Since $B_{\alpha\beta}$ fully symmetric

$$B_{\alpha\beta} = B_2 o_\alpha o_\beta + 2B_1 o_{(\alpha} \iota_{\beta)} + B_0 \iota_\alpha \iota_\beta$$

where $\iota_\alpha = (0, 1)$ and $o_\alpha = (1, 0)$.

- Asymptotically, the field satisfies a peeling property

$$B_n = \frac{B_n^0}{r^{3-n}} + \cdots$$

with r being the Bondi radial coordinate.

- Define $R_n = B_{1-n}^0$, then the asymptotic equations become

$$\boxed{\partial_u R_n + \partial_{\bar{z}} R_{n-1} + [\bar{\mathcal{A}}, R_{n-1}] = 0}$$

where $A|_{\mathscr{I}} = \bar{\mathcal{A}} d\bar{z}$.

- Extending these to all $n \geq -1$ gives the recursion relations consider by Freidel, Pranzetti, and Raclariu.

Recursion Relations

- The $R_{n>1}$ are seen as subleading orders of B_0

$$B_0 = \frac{B_0^0}{r^3} + \sum_{n=1}^{\infty} \frac{\partial_z^n R_{n+1} + \Phi_n}{n! r^{3+n}}$$

- Using the integrability of SDYM, we can construct recursion relations for $\bar{\mathcal{A}}$

$$\boxed{\partial_u \tilde{R}_n + \partial_{\bar{z}} \tilde{R}_{n-1} + [\bar{\mathcal{A}}, \tilde{R}_{n-1}] = 0}$$

with $\tilde{R}_{-1} = \bar{\mathcal{A}}$.

- The Lax pair commuting $[L_{\dot{\alpha}}, L_{\dot{\beta}}] = 0$ is a consistency condition for the existence of a matrix-valued function satisfying

$$L_{\dot{\alpha}} T = 0 \quad \text{with} \quad T = T(x, q)$$

- Considering the large q expansion

$$T \partial_u T^{-1} = \sum_{n=0}^{\infty} \frac{\tilde{R}_{n-1}}{q^{n+1}}$$

we find the $\tilde{R}_s$.

Twistor Space

Twistor Theory Basics

- The twistor space of the complexified Minkowski space is given by

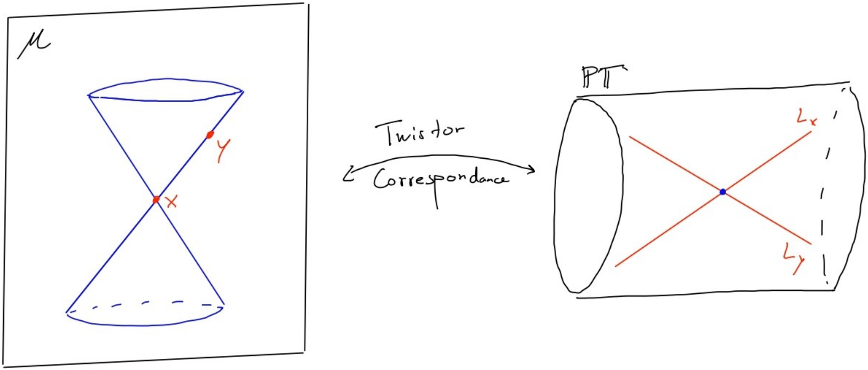
$$\begin{aligned}\mathbb{PT} &= \mathcal{O}(1) \oplus \mathcal{O}(1) \to \mathbb{CP}^1 \\ &= \mathbb{CP}^3 - \mathbb{CP}^1_{\lambda_\alpha = 0}\end{aligned}$$

  with homogeneous coordinates $Z^A = [\mu^{\dot{\alpha}}, \lambda_\alpha] \in \mathbb{CP}^3$.
- Spacetime is the moduli space of holomorphic curves. The homolorphic lines $L \simeq \mathbb{CP}^1$ are given by the incidence relations

$$\mu^{\dot{\alpha}} = i x^{\alpha\dot{\alpha}} \lambda_\alpha \qquad x^{\alpha\dot{\alpha}} = \begin{pmatrix} u + rz\bar{z} & -rz \\ -r\bar{z} & r \end{pmatrix}$$

  with $x^{\alpha\dot{\alpha}}$ labeling the coordinates on spacetime.
- Nonlocal relationship between spacetime and twistor space. A point x in spacetime is a holomorphic line L_x in twistor space.

Twistor Theory Basics



Fact: Two twistor lines intersect if the corresponding spacetime points are null separated.

Infinity on Twistor Space

- The boundary of Minkowski space can be seen on twistor space:

$$\mathbb{CP}^1_{\lambda_\alpha=0} = i^0 \quad \mathscr{I} = \text{All lines interesecting } \mathbb{CP}^1_{\lambda_\alpha=0}$$

- From the incidence relations, we can see this relationship

$$\lambda_\alpha = \frac{x_{\alpha\dot{\alpha}}}{ix^2}\mu^{\dot{\alpha}} \xrightarrow{r=\infty} n\zeta_\alpha \qquad \zeta_\alpha = \begin{pmatrix} 1 \\ z \end{pmatrix}$$

with $n \sim u^{-1}$.

- The twistor space over $\mathscr{I}_\mathbb{C}$ is given by the blow-up

$$\overline{\mathbb{PT}} = \{([Z^A], [\zeta_\alpha]) \in \mathbb{CP}^3 \times \mathbb{CP}^1 | \langle\lambda\zeta\rangle = 0\}$$

- There exists a projection to $\mathscr{I}_\mathbb{C}$

$$p : \overline{\mathbb{PT}} \to \mathscr{I}_\mathbb{C} \quad \text{with} \quad u = \frac{[\mu\bar{\lambda}]}{\lambda_0\bar{\lambda}_{\dot{0}}}, \; z = \frac{\lambda_1}{\lambda_0},$$

with the fiber coordinate is given by $q = \mu^{\dot{1}}/\lambda_0$.

SDYM on Twistor Space

SDYM appears on twistor space through the Ward correspondence

Theorem (Ward 1977)

There is a one-to-one correspondance between

1. SD gauge fields on $\mathcal{M}$, and
2. holomorphic vector bundles $E \to \mathbb{PT}$ with $E|_L$ trivial for every twistor line $L \simeq \mathbb{CP}^1$ corresponding to $x \in \mathcal{M}$.

Holomorphicity of E is expressed as the integrability of the Dolbeault operator

$$\bar{D} = \bar{\partial} + a \quad , \quad a \in \Omega^{0,1}(\mathbb{PT}, \mathfrak{g} \otimes \mathcal{O})$$

giving

$$\bar{D}^2 = 0 = \bar{\partial} a + \frac{1}{2}[a, a]$$

Penrose Transform

- The Penrose transform (Penrose 1969) relates cohomology data on twistor space to solutions of PDEs on spacetime. In particular,

$$b \in H^{0,1}_{\bar{D}}(\mathbb{PT}, \mathcal{O}(-4) \otimes \mathrm{End}(E)) \cong \{B_{\alpha\beta} \text{ on } \mathcal{M} | D^{\alpha\dot{\alpha}} B_{\alpha\beta} = 0\}$$

- There exists an integral formula for this correspondence

$$B_{\alpha\beta} = \frac{1}{2\pi i} \int_{L_x} D\lambda \wedge \lambda_\alpha \lambda_\beta \; f \, b \, f^{-1}|_{L_x}$$

- The bundle $E \to \mathbb{PT}$ being (holomorphically) trivial allows for the existence of $f : E|_L \to \mathbb{C}^r$ such that

$$\bar{\partial}|_L f + a|_L f = 0$$

From f one can reconstruct the SD gauge field $A_{\alpha\dot{\alpha}}$ (Ward 1977, Sparling 1990).

Radiative Data on Twistor Space

- Using the projection from asymptotic twistor space to null infinity $p : \overline{\mathbb{PT}} \to \mathscr{I}_{\mathbb{C}}$, we can pull back the radiative data

$$a = p^*(\bar{\mathcal{A}}(u, z, \bar{z})d\bar{z}) = \bar{\mathcal{A}}([\mu\bar{\lambda}], \lambda, \bar{\lambda})D\bar{\lambda}$$

  to obtain the partial connection.
- We work with a partial connection gauge equivalent to the radiative data

$$f^{-1}(\bar{\partial} + a)f = \bar{\mathcal{A}}D\bar{\lambda}$$

- The rest of the asymptotic data appears via integral formulae.

Twistor Action and Symmetries

- Self-dual Yang-Mills + linearized perturbation:

$$S[a,b] = \int_{\mathbb{PT}} D^3Z \wedge \mathrm{tr}\left(b \wedge \left(\bar{\partial} a + \frac{1}{2}[a,a]\right)\right)$$

with fields $a \in \Omega^{0,1}(\mathbb{PT}, \mathcal{O})$ and $b \in \Omega^{0,1}(\mathbb{PT}, \mathcal{O}(-4))$.

- The equations of motion and symmetries of the action

$$\bar{\partial} a + \frac{1}{2}[a,a] = 0 \qquad \delta_\xi a = \bar{D}\xi$$

$$\bar{D} b = 0 \qquad \delta_{\xi,\phi} b = [b,\xi] + \bar{D}\phi$$

with $\xi \in \Omega^0(\mathbb{PT}, \mathcal{O})$ and $\phi \in \Omega^0(\mathbb{PT}, \mathcal{O}(-4))$.

- The twistor action can be shown to reproduce the Chalmers-Siegel action (Chalmers, Siegel 1996)

$$S[B,A] = \int_{\mathcal{M}} d^4x \mathrm{Tr}\left(B_{\alpha\beta} F^{\alpha\beta}\right)$$

Twistor Charges

- The symplectic structure coming from the action is given by

$$\Omega[\delta_1, \delta_2] = \int_\Sigma D^3 Z \wedge \mathrm{Tr}\left(\delta_1 b \wedge \delta_2 a - \delta_2 b \wedge \delta_1 a\right)$$

  where Σ is a codimension 1 hypersurface.

- By contracting the symmetries of the action into the symplectic structure, we obtain the charges

$$\delta_\alpha \lrcorner \Omega = \delta H_\alpha - H_{\delta\alpha} \qquad , \qquad H_\alpha = \int_{\partial\Sigma} D^3 Z \wedge \mathrm{Tr}(\xi b + \phi \partial_u a)$$

  where $\alpha = (\xi, \phi)$ The charges are non-integrable due to the field dependence of the parameters.

Charge Algebra

- The non-integrable charges still satisfy an algebra when an appropriate bracket is used (Barnich, Troessaert '11)

$$\{H_\alpha, H_{\alpha'}\}_\star = \delta_\alpha H_{\alpha'} - H_{\delta_{\alpha'}\alpha} = H_{[\alpha,\alpha']_\star} + K_{\alpha,\alpha'}$$

- We obtain a representation of the algebroid bracket

$$[\alpha, \alpha']_\star = ([\xi, \xi'] + \delta_\alpha \xi' - \delta_{\alpha'}\xi, [\phi, \xi'] + [\xi, \phi'] + \delta_\alpha \phi' - \delta_{\alpha'}\phi)$$

- We also find a cocyle $K_{\alpha,\alpha'}$ satisfying

$$K_{[\alpha_1,\alpha_2]_\star,\alpha_3} - \delta_{\alpha_3} K_{\alpha_1,\alpha_2} + \text{cyclic}(1,2,3) = 0$$

Celestial Currents

Asymptotic Integral Formulae

- To translate the charges to spacetime, we must find integral formulae solving the recursion relations

$$\partial_u R_s + \partial_{\bar{z}} R_{s-1} + [\bar{\mathcal{A}}, R_{s-1}] = 0$$
$$\partial_u \tilde{R}_s + \partial_{\bar{z}} \tilde{R}_{s-1} + [\bar{\mathcal{A}}, \tilde{R}_{s-1}] = 0$$

- This is achieved by the asymptotic twistor integral formulae

$$R_s = \frac{1}{2\pi i} \int_{L_{u,z}} q^{s+1} dq \wedge f^{-1}\, bf|_{L_{u,z}}$$
$$\tilde{R}_s = \frac{1}{2\pi i} \int_{L_{u,z}} q^{s+1} dq \wedge f^{-1}\, \partial_u a f|_{L_{u,z}}$$

when a, b satisfy their equations of motion.

Gauge Parameters

- If ξ, ϕ are global functions on twistor space, the gauge symmetries on twistor space give vanishing Hamiltonian charges. These are small gauge transformations.
- The large gauge transformations are the ones that break the boundary conditions
$$a \sim O(u^{-1}) \qquad b \sim O(u^3)$$
- We obtain Celestial currents by choosing the gauge parameter to satisfy
$$\bar{D}(\xi, \phi) = 0$$
on $\mathbb{PT} - \{z = 0, \infty\}$. On non-trivial backgrounds, the gauge parameters are field-dependent.

Gauge Parameters

- First let us consider a trivial background $a = 0$.
- For example

$$\xi_{s=k+l} = \sum_{n\in\mathbb{Z}} \sum_{k+l=s} L^a_{k,l,n} \xi^a_{k,l,n} \qquad \xi^a_{k,l,n} = \frac{(\mu^{\dot{0}})^k (\mu^{\dot{1}})^l}{\lambda_0^{k+l+n} \lambda_1^{-n}} t^a$$

  where $\xi_s \sim O(u^s)$ breaks the boundary condition of $a \sim O(u^{-1})$. Similar expressions hold for ϕ_s on a trivial background.
- Consider the algebra between these generators

$$[\xi_{k,l,n}, \xi_{k',l',n'}] = f_c^{ab} \xi_{k+k',l+l',n+n'}$$
$$[\xi_{k,l,n}, \phi_{k',l',n'}] = f_c^{ab} \phi_{k+k',l+l',n+n'}$$

Gauge Parameters

- On a non-trivial background, use the frame to solve the gauge parameter equation

$$\xi = f^{-1}\tilde{\xi}f \qquad , \qquad \phi = f^{-1}\tilde{\phi}f$$

- Expanding in a power series in q

$$\tilde{\xi} = \sum_{n=0}^{\infty} \xi_n q^n \qquad , \qquad \tilde{\phi} = \sum_{n=0}^{\infty} \phi_n q^n$$

- The condition on the gauge parameter then gives the dual recursion relations

$$\partial_u \xi_{n-1} + \partial_{\bar{z}} \xi_n + [\bar{\mathcal{A}}, \xi_n] = 0$$
$$\partial_u \phi_{n-1} + \partial_{\bar{z}} \phi_n + [\bar{\mathcal{A}}, \phi_n] = 0$$

- The first are the dual recursion relations found by (Cresto, Friedel'25).

Celestial Currents

- We use $\Sigma = \{|z| = 1\}$ and the $\bar{D}(\xi, \phi) = 0$ gauge parameters, the Hamiltonian charges are modes of currents

$$H_\xi = \frac{1}{2\pi i} \oint_{|z|=1} dz\, J_\xi \qquad , \qquad \tilde{H}_\phi = \frac{1}{2\pi i} \oint_{|z|=1} dz\, \tilde{J}_\phi$$

- The currents are given by

$$J_\xi(z) = \sum_{n=0}^{\infty} \oint_{u=\infty} du \operatorname{Tr}\left(\xi_n R_{n-1}\right) \qquad , \qquad \partial_{\bar{z}} J_\xi = 0$$

$$\tilde{J}_\phi(z) = \sum_{n=0}^{\infty} \oint_{u=\infty} du \operatorname{Tr}\left(\phi_n \tilde{R}_{n-1}\right) \qquad , \qquad \partial_{\bar{z}} \tilde{J}_\phi = 0$$

where holomorphicity is a consequence of the recursion relations.

Celestial Currents

- The generators ξ, ϕ are seeded by the trivial background result

$$\xi^a_{k,l,n} = \frac{(\mu^{\dot{0}})^k (\mu^{\dot{1}})^l}{\lambda_0^{k+l+n} \lambda_1^{-n}} t^a + O(\bar{\mathcal{A}})$$

$$\phi^a_{k,l,n} = \frac{(\mu^{\dot{0}})^k (\mu^{\dot{1}})^l}{\lambda_0^{k+l+n+4} \lambda_1^{-n}} t^a + O(\bar{\mathcal{A}})$$

- The currents take the form

$$J^a[k,l,m] = \frac{1}{2\pi i} \oint_{|z|=1} dz z^m J^a[k,l](z)$$

$$J^a[k,l] = \sum_{n=0}^{k} \frac{k!}{n!(k-n)!} \oint_{u=\infty} du u^{k-m} (-\bar{z})^n R^a_{l+n-1} + O(\bar{\mathcal{A}})$$

with similar expressions holding for $\tilde{J}^a[k,l,m]$.

- At a linearized level, reproduces expected results (Pano, Puhm, Trevisani 23')

Celestial Algebra

- Using the algebroid bracket, we can show

$$[\xi^a_{k,l,m}, \xi^b_{k',l',m'}]_\star = f^{ab}_c \xi^c_{k+k',l+l',m+m'}$$
$$[\xi^a_{k,l,m}, \phi^b_{k',l',m'}]_\star = f^{ab}_c \phi^c_{k+k',l+l',m+m'}$$

- Since the Hamiltonian charges are a representation of these brackets, we find

$$\{J^a[k,l,m], J^b[k',l',m']\}_\star = f^{ab}_c J[k+k', l+l', m+m']$$
$$\{J^a[k,l,m], \tilde{J}^b[k',l',m']\}_\star = f^{ab}_c \tilde{J}[k+k', l+l', m+m']$$
$$\{\tilde{J}^a[k,l,m], \tilde{J}^b[k',l',m']\}_\star = 0$$

Spacetime Charges

Spacetime Charges

- We now wish to find charges at a finite cut of $\mathscr{I}$
- To pick out real $\mathscr{I}$ use $\Sigma = \{u = \bar{u}\}$. Furthermore, consider the gauge parameters that satisfy $\partial_q \bar{D}(\xi, \phi) = 0$ instead.
- This condition comes from the fact that $a \sim \bar{\mathcal{A}} d\bar{\lambda}$ through the frame.
- The $a = 0$ result coincides with the other gauge parameter choice

$$\xi_s = \frac{1}{s!} \frac{\partial^s \rho_s(\lambda, \bar{\lambda})}{\partial \bar{\lambda}^{\dot{\alpha}_1} \dots \partial \bar{\lambda}^{\dot{\alpha}_s}} \mu^{\dot{\alpha}_1} \cdots \mu^{\dot{\alpha}_s} = \sum_{n \in \mathbb{Z}} \sum_{k+l=s} L^a_{k,l,n} \xi^a_{k,l,n}$$

- These gauge parameters satisfy the S-algebra in a slightly different form

$$[\xi_s(\rho_s), \xi_{s'}(\rho'_s)] = \xi_{s+s'}([\rho_s, \rho_{s'}])$$

Spacetime Charges

- For a non-trivial background, once again use the frame to obtain the dual recursion relations.
- The charges at a finite cut of $\mathscr{I}$

$$H_\xi = \sum_{n=0}^{\infty} \int_{\mathbb{CP}^1} dz d\bar{z} \mathrm{Tr}(\xi_n R_n) \qquad \tilde{H}_\phi = \sum_{n=0}^{\infty} \int_{\mathbb{CP}^1} dz d\bar{z} \mathrm{Tr}(\phi_n \tilde{R}_n)$$

- These charges have the property that they are conserved when $R_{-1} = 0 = \tilde{R}_{-1}$

$$\partial_u H_\xi = \int_{\mathbb{CP}^1} dz d\bar{z} \mathrm{Tr}\left[R_{-1}\left(\partial_{\bar{z}}\xi_0 + [\bar{\mathcal{A}}, \xi_0]\right)\right]$$
$$\partial_u \tilde{H}_\phi = \int_{\mathbb{CP}^1} dz d\bar{z} \mathrm{Tr}\left[\tilde{R}_{-1}\left(\partial_{\bar{z}}\phi_0 + [\bar{\mathcal{A}}, \phi_0]\right)\right]$$

where we use the recursion relations and gauge parameter condition.

Symmetries on Spacetime

- Using the natural symplectic structure coming from twistor space

$$\Omega = \int D^3 Z \wedge \mathrm{Tr}(\delta b \wedge \delta a) = \int_{\mathscr{I}} du dz d\bar{z} \mathrm{Tr}(\delta R_{-1} \delta \bar{A})$$

- We find the asymptotic data transform as

$$\delta_\alpha R_n = \sum_{m=0}^{\infty} [R_{n+m}, \xi_m] + \omega_n(\phi)$$

$$\delta_\alpha \tilde{R}_n = \sum_{m=0}^{\infty} [\tilde{R}_{n+m}, \xi_m] + \tilde{\omega}_n(\xi)$$

  These variations are symmetries of the recursion relations.

- The algebra of the variations manifests the algebroid bracket

$$[\delta_\xi, \delta_{\xi'}] = \delta_{[\xi,\xi']_\star} \quad , \quad [\delta_\xi, \delta_\phi] = \delta_{[\xi,\phi]_\star} \quad , \quad [\delta_\phi, \delta_{\phi'}] = 0$$

Symmetries on Spacetime

- What do the variations look like?
- The first few ξ_s transformations are given by

$$
\begin{aligned}
\delta_{\xi_0} R_n &= [R_n, \rho_0] \\
\delta_{\xi_1} R_n &= [R_{n+1}, \rho_1] - u[R_n, \partial_{\bar{z}}\rho_1] - [R_n, [\partial_u^{-1}\bar{\mathcal{A}}, \rho_1]] \\
\delta_{\xi_2} R_s &= [R_{s+2}, \rho_2] - u[R_{s+1}, \partial_{\bar{z}}\rho_2] - [R_{s+1}, [\partial_u^{-1}\bar{\mathcal{A}}, \rho_2]] + \frac{u^2}{2}[R_s, \partial_{\bar{z}}^2\rho_2] \\
&\quad + [R_s, \partial_{\bar{z}}[\partial_u^{-2}\bar{\mathcal{A}}, \rho_2]] + [R_s, [\partial_u^{-1}(u\bar{\mathcal{A}}), \partial_{\bar{z}}\rho_2]] + \cdots
\end{aligned}
$$

  where $\rho_s = \rho_s(z, \bar{z})$ satisfying $\partial_{\bar{z}}^{s+1}\rho_s = 0$ the wedge condition.
- The first few ϕ transformations are given by

$$
\delta_{\phi_0} R_{-1} = [\partial_u^{-1}\bar{\mathcal{A}}, \chi_0] \qquad \delta_{\phi_0} R_0 = [\chi_0, \partial_{\bar{z}}\partial_u^{-2}\bar{\mathcal{A}}] - \partial_u^{-1}[\bar{\mathcal{A}}, [\partial_u^{-1}\bar{\mathcal{A}}, \chi_0]]
$$

- Similar expressions hold for the variations of $\tilde{R}_n$.

Conclusion and Future Directions

Conclusion:

- The Ward construction for SDYM allows the study of the S-algebra locally on twistor space.
- We have considered the S-algebra from both the phase space perspective and the Celestial CFT perspective.
- The analysis has been done around a nontrivial self-dual background.

Future Directions:

- Can extend these ideas to SD gravity. (A.K., Mason, Ruzziconi, Srikant '24)
- The S-algebra (Mason, Woodhouse 1996) and LHam($\mathbb{C}^2$) (Dunajski, Mason 00') can be seen in the bulk of spacetime
- In gravity, what do the charges look like with $\Lambda \neq 0$?

Thank you for listening!