GEOMETRICAL CONSTRAINTS IN THE LEVEL SET METHOD FOR SHAPE AND TOPOLOGY OPTIMIZATION

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Results obtained in collaboration with F. Jouve (LJLL, Paris 7), G. Michailidis (formerly Renault and CMAP).

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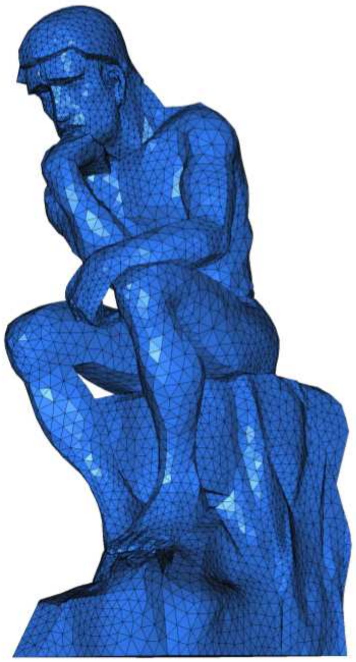
RODIN project



Ecole Polytechnique,

UPMC, INRIA,

Renault, EADS,

ESI group, etc.

1. Introduction.
2. Shape and topology optimization by the level set method.
3. Geometrical constraints.
4. Numerical results.
5. Molding constraints.

-I- INTRODUCTION

Shape optimization : minimize an objective function over a set $\mathcal{U}_{ad}$ of admissibles shapes Ω (including possible constraints)

$$\inf_{\Omega \in \mathcal{U}_{ad}} J(\Omega)$$

The objective function is evaluated through a partial differential equation (state equation)

$$J(\Omega) = \int_\Omega j(u_\Omega)\, dx$$

where u_Ω is the solution of

$$PDE(u_\Omega) = 0 \quad \text{in} \quad \Omega$$

Topology optimization : the optimal topology is unknown.

The model of linear elasticity

A shape is an open set $\Omega \subset \mathbb{R}^d$ with boundary $\partial\Omega = \Gamma \cup \Gamma_N \cup \Gamma_D$.

For a given applied load $g : \Gamma_N \to \mathbb{R}^d$, the displacement $u : \Omega \to \mathbb{R}^d$ is the solution of

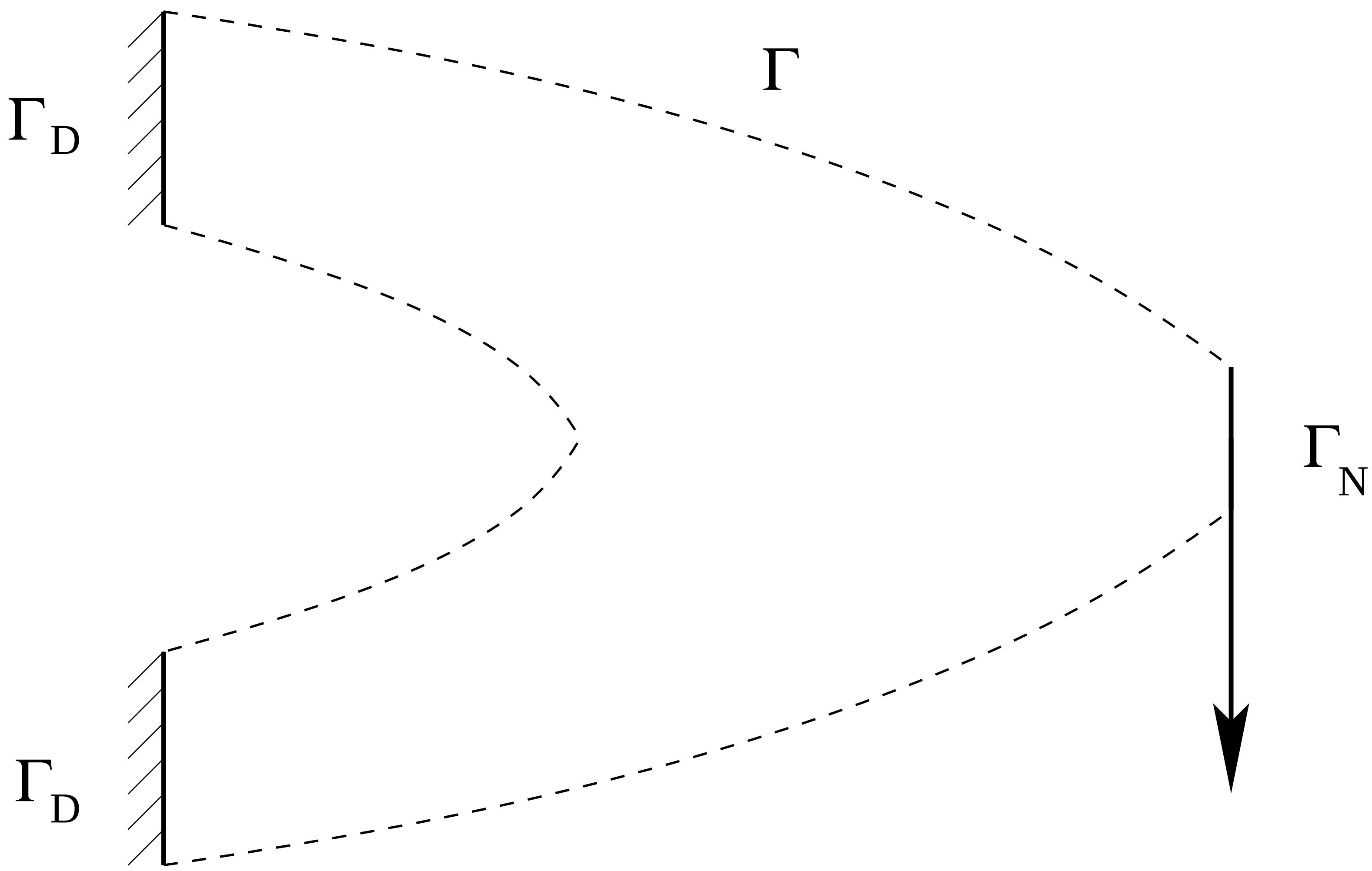
$$\begin{cases} -\operatorname{div}(A\,e(u)) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \\ \big(A\,e(u)\big)n = g & \text{on } \Gamma_N \\ \big(A\,e(u)\big)n = 0 & \text{on } \Gamma \end{cases}$$

with the strain tensor $e(u) = \frac{1}{2}\left(\nabla u + \nabla^t u\right)$, the stress tensor $\sigma = Ae(u)$, and A an homogeneous isotropic elasticity tensor.

Typical objective function: the compliance

$$J(\Omega) = \int_{\Gamma_N} g \cdot u\, dx,$$

Example: the cantilever



Admissible shapes

The **shape optimization** problem is

$$\inf_{\Omega \in \mathcal{U}_{ad}} J(\Omega),$$

where the set of admissible shapes is typically

$$\mathcal{U}_{ad} = \left\{ \Omega \subset D \text{ open set such that } \Gamma_D \bigcup \Gamma_N \subset \partial\Omega \text{ and } \int_\Omega dx = V_0 \right\},$$

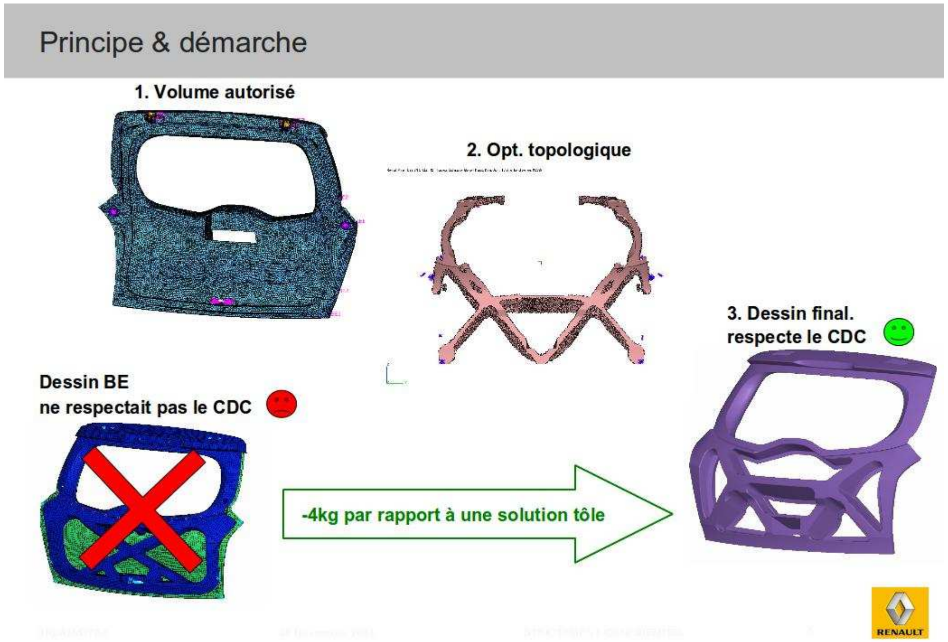
where $D \subset \mathbb{R}^d$ is a given "working domain" and V_0 is a prescribed volume.

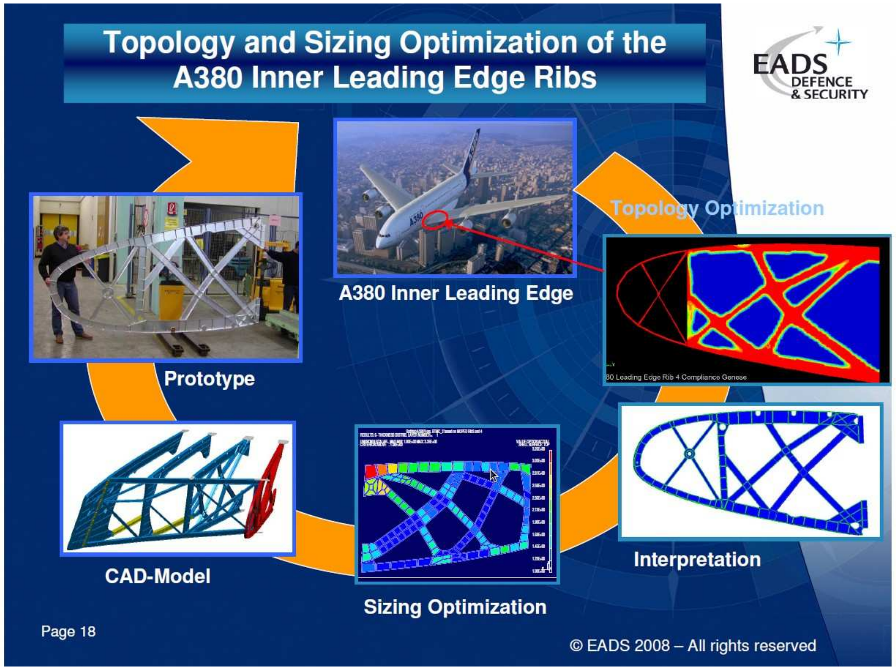
☞ The boundary subsets Γ_D and Γ_N are fixed. Only Γ is optimized (free boundary).

☞ Existence of optimal shapes is a delicate issue (typically, one needs further constraints in $\mathcal{U}_{ad}$).

☞ A nice numerical method is the level set algorithm since it allows for topology changes.

Industrial applications





- ☞ Tremendous progresses were achieved on academic research about shape and topology optimization.
- ☞ There are several commercial softwares used by industry.
- ☞ But manufacturability of the optimal shapes is not always guaranteed.

Goal of the present work

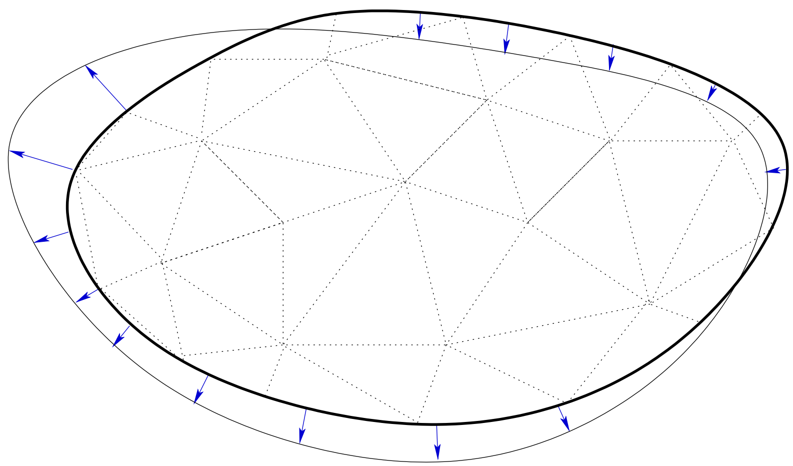
- ☞ We want to add geometrical constraints (for manufacturability), i.e., constraints on Ω, not on the state u_Ω.
- ☞ The level set framework is well suited for this because it relies on the distance function to the boundary.
- ☞ Issues to be addressed concerning geometrical constraints: modelling, shape differentiation, numerical implementation.

Before that, let's review the state of the art about the level set method for shape and topology optimization.

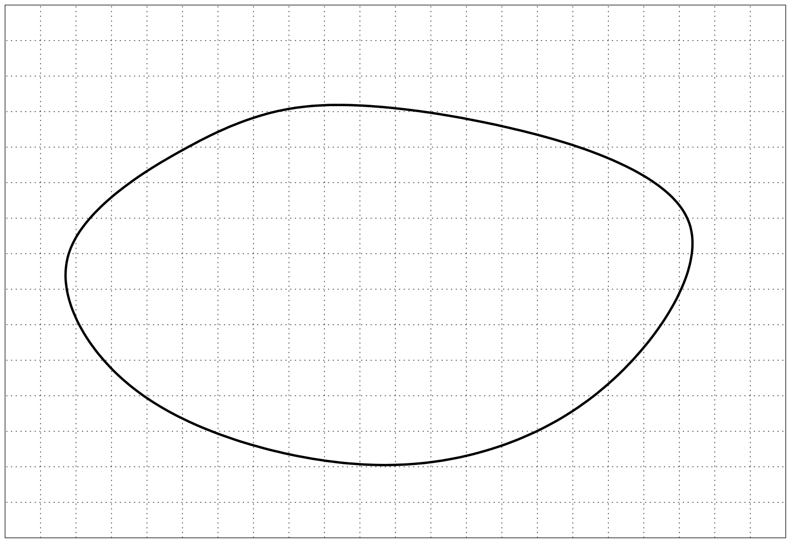
-II- LEVEL SET METHOD

A new numerical implementation of an old idea...

- ☞ Framework of Hadamard's method of shape variations.
- ☞ Main tool: the level set method of Osher and Sethian (JCP 1988).
- ☞ Shape capturing algorithm.
- ☞ Fixed mesh: low computational cost.
- ☞ Early references: Sethian and Wiegmann (JCP 2000), Osher and Santosa (JCP 2001), Allaire, Jouve and Toader (CRAS 2002, JCP 2004, CMAME 2005), Wang, Wang and Guo (CMAME 2003).



Shape tracking



Shape capturing

FRONT PROPAGATION BY LEVEL SET

Shape capturing method on a fixed mesh of the "working domain" D.

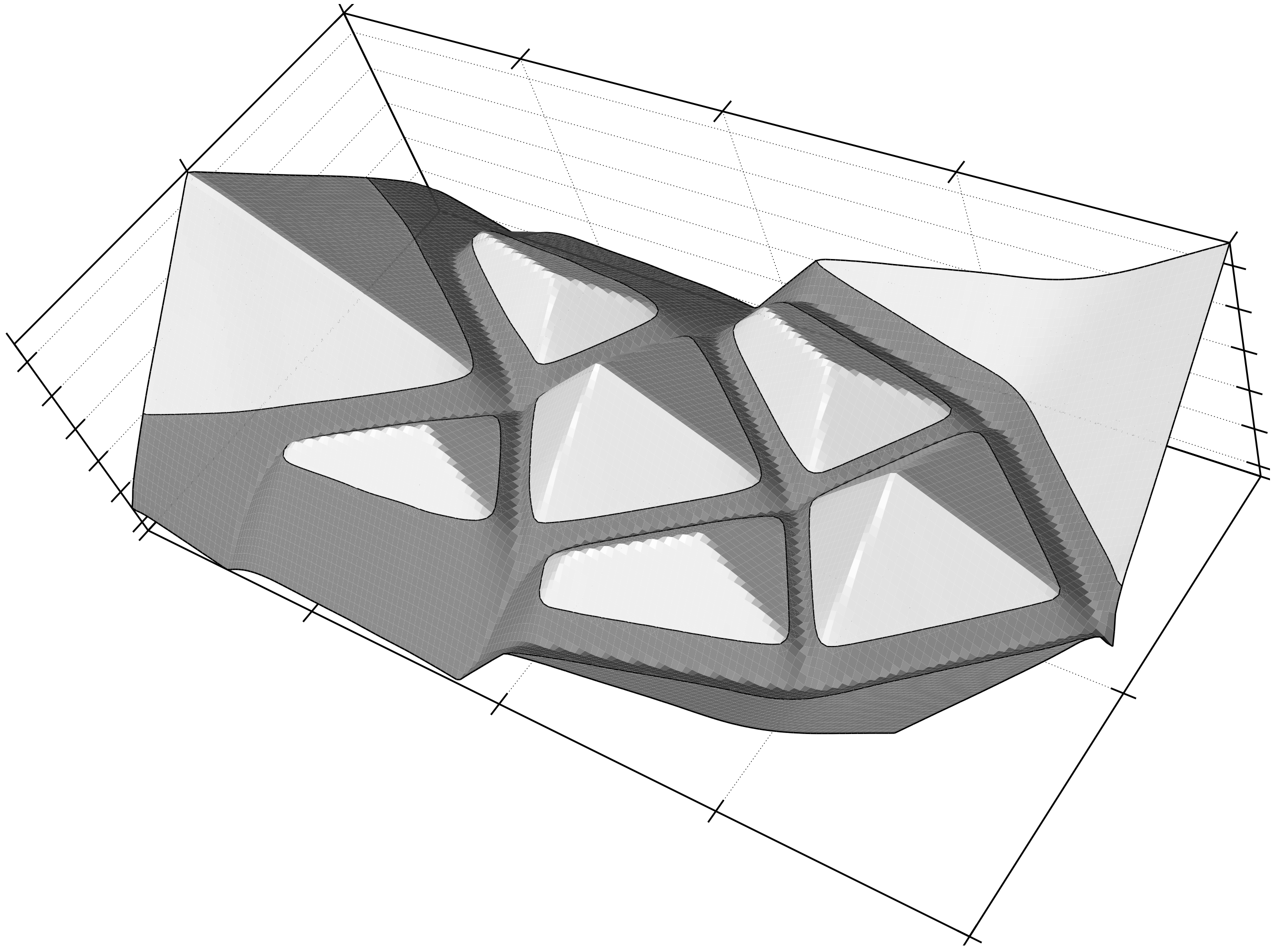
A shape Ω is parametrized by a **level set** function

$$\begin{cases} \psi(x) = 0 & \Leftrightarrow x \in \partial\Omega \cap D \\ \psi(x) < 0 & \Leftrightarrow x \in \Omega \\ \psi(x) > 0 & \Leftrightarrow x \in (D \setminus \Omega) \end{cases}$$

Assume that the shape $\Omega(t)$ evolves in time t with a normal velocity $V(t, x)$. Then its motion is governed by the following Hamilton Jacobi equation

$$\frac{\partial \psi}{\partial t} + V|\nabla_x \psi| = 0 \quad \text{in } D.$$

Example of a level set function



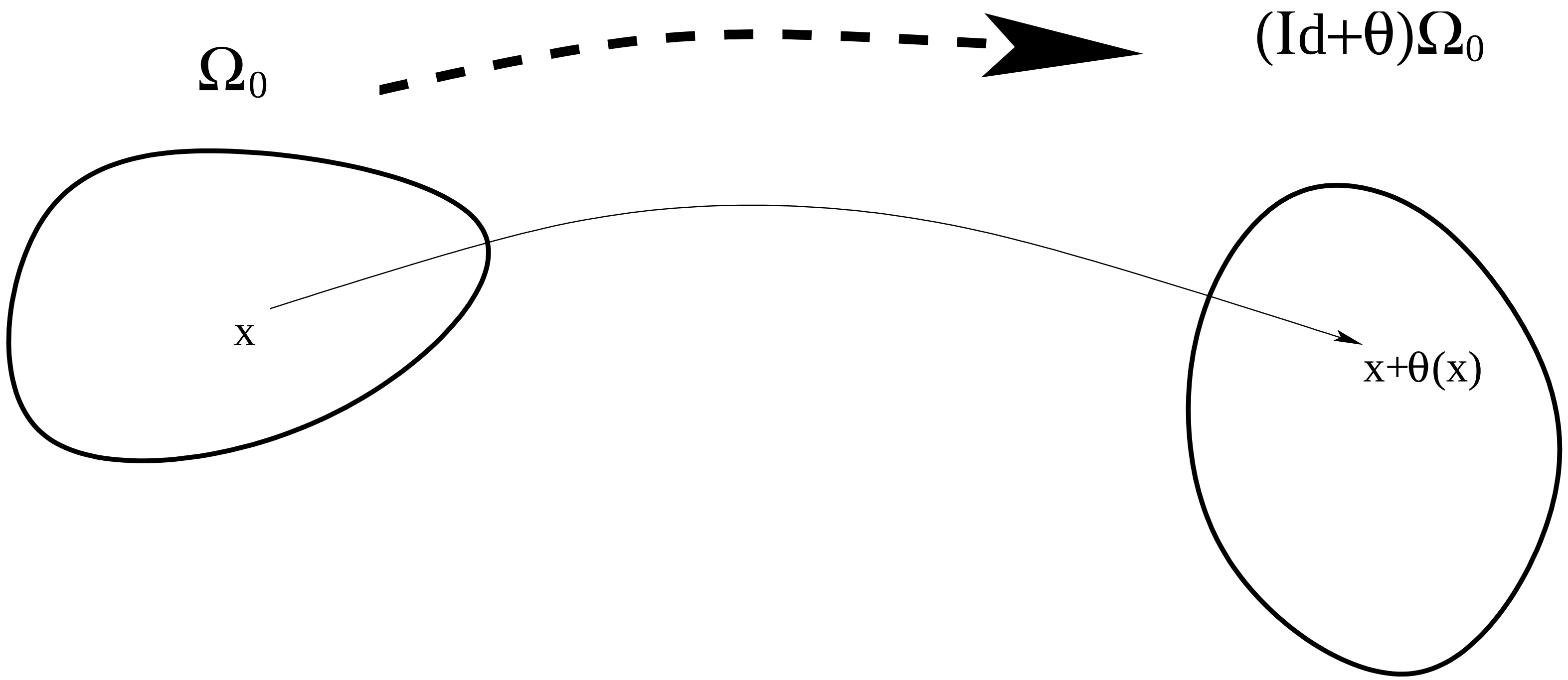
Advection velocity = shape gradient

The velocity V is deduced from the shape gradient of the objective function.

To compute this shape gradient we recall the well-known Hadamard's method.

Let Ω_0 be a reference domain. Shapes are parametrized by a vector field θ

$$\Omega = (\,\mathrm{Id} + \theta)\Omega_0 \quad \text{with} \quad \theta \in C^1(\mathbb{R}^d;\mathbb{R}^d).$$



Shape derivative

Definition: the shape derivative of $J(\Omega)$ at Ω_0 is the Fréchet differential of $\theta \to J\big((\mathrm{Id} + \theta)\Omega_0\big)$ at 0.

Hadamard structure theorem: the shape derivative of $J(\Omega)$ can always be written (in a distributional sense)

$$J'(\Omega_0)(\theta) = \int_{\partial\Omega_0} \theta(x) \cdot n(x)\, j(x)\, ds$$

where $j(x)$ is an integrand depending on the state u and an adjoint p.

We choose the velocity $V = \theta \cdot n$ such that $J'(\Omega_0)(\theta) \leq 0$.

Simplest choice: $V = \theta \cdot n = -j$ but other ones are possible (including regularization).

SHAPE DERIVATIVE OF THE COMPLIANCE

$$J(\Omega) = \int_{\Gamma_N} g \cdot u_\Omega \, ds = \int_\Omega A\, e(u_\Omega) \cdot e(u_\Omega)\, dx,$$

where u_Ω is the state variable in Ω.

$$J'(\Omega)(\theta) = -\int_\Gamma Ae(u_\Omega) \cdot e(u_\Omega)\, \theta \cdot n\, ds,$$

Remarks:

1. self-adjoint problem (no adjoint state is required),
2. taking into account the volume constraint add a fixed Lagrange multiplier $\lambda - Ae(u_\Omega) \cdot e(u_\Omega)$.

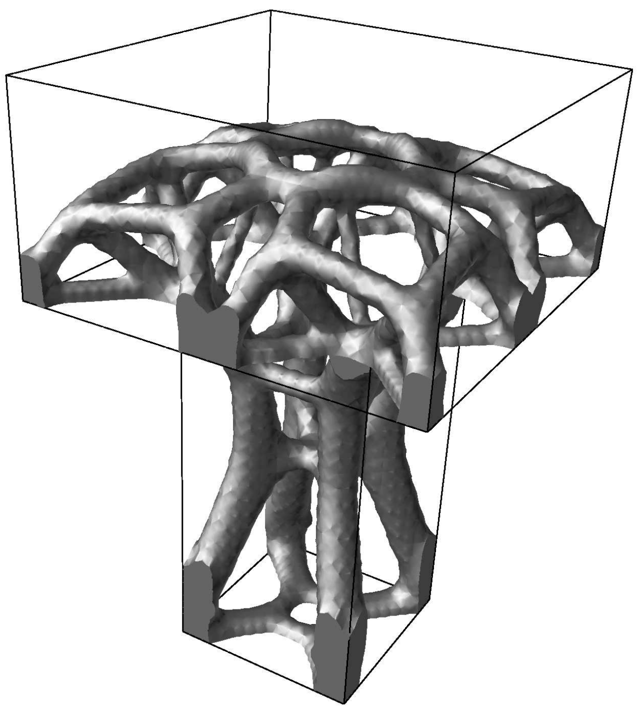
NUMERICAL ALGORITHM

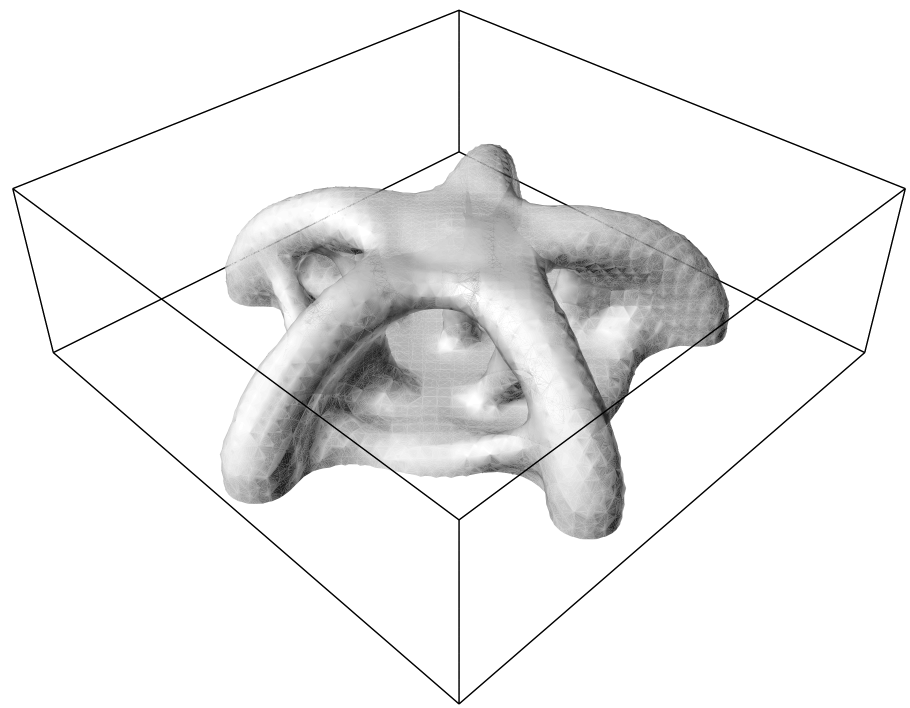
1. Initialization of the level set function ψ_0 (including holes).
2. Iteration until convergence for $k \geq 1$:
   (a) Compute the elastic displacement u_k for the shape ψ_k.
   Deduce the shape gradient = normal velocity = V_k
   (b) Advect the shape with V_k (solving the Hamilton Jacobi equation) to obtain a new shape ψ_{k+1}.

For numerical examples, see the web page:

http://www.cmap.polytechnique.fr/~optopo/level_en.html

Examples of results with complex topologies





-III- GEOMETRICAL CONSTRAINTS

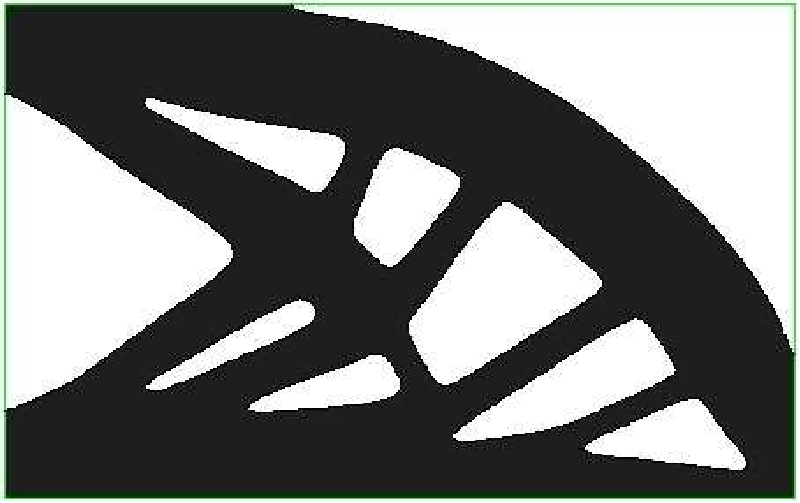
We focus on thickness control because of

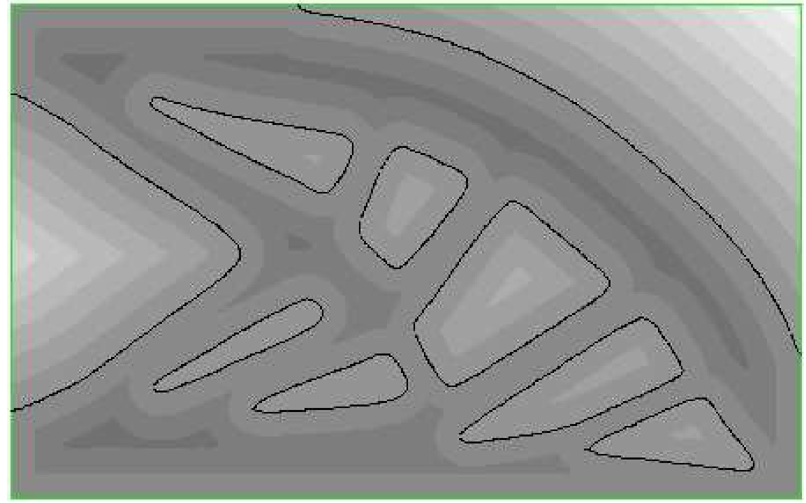
- manufacturability,
- uncertainty in the microscale (MEMS design),
- robust design (fatigue, buckling, etc.).

Existing works:

- Several approaches in the framework of the **SIMP** method to ensure minimum length scale (Sigmund, Poulsen, Guest, etc.).
- In the **level-set** framework: Chen, Wang and Liu implictly control the feature size by adding a "line" energy term to the objective function ; Alexandrov and Santosa kept a fixed topology by using offset sets.
- Many works in **image processing**.

Signed-distance function





Definition. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain. The **signed distance function** to Ω is the function $\mathbb{R}^d \ni x \mapsto d_\Omega(x)$ defined by :

$$d_\Omega(x) = \begin{cases} -d(x, \partial\Omega) & \text{if } x \in \Omega \\ 0 & \text{if } x \in \partial\Omega \\ d(x, \partial\Omega) & \text{if } x \in \mathbb{R}^d \setminus \Omega \end{cases}$$

where $d(\cdot, \partial\Omega)$ is the usual Euclidean distance.

Constraint formulations

Maximum thickness.

Let $d_{\max}$ be the maximum allowed thickness. The constraint reads:

$$d_\Omega(x) \geq -d_{\max}/2 \quad \forall x \in \Omega$$

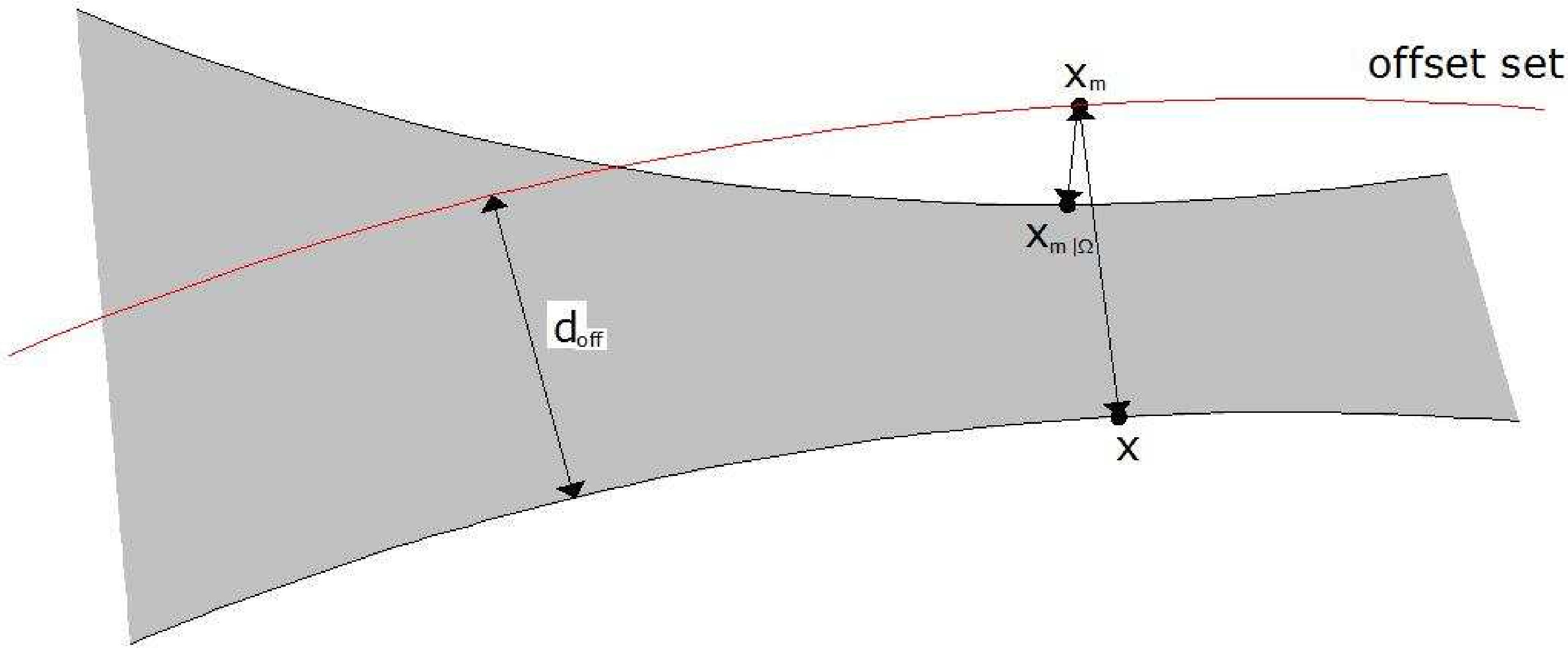
Minimum thickness

Let $d_{\min}$ be the minimum allowed thickness. The constraint reads:

$$d_\Omega(x - d_{\text{off}} n(x)) \leq 0 \quad \forall x \in \partial\Omega,\ \forall d_{\text{off}} \in [0, d_{\min}]$$

Remark: similar constraints for the thickness of holes.

Offset sets

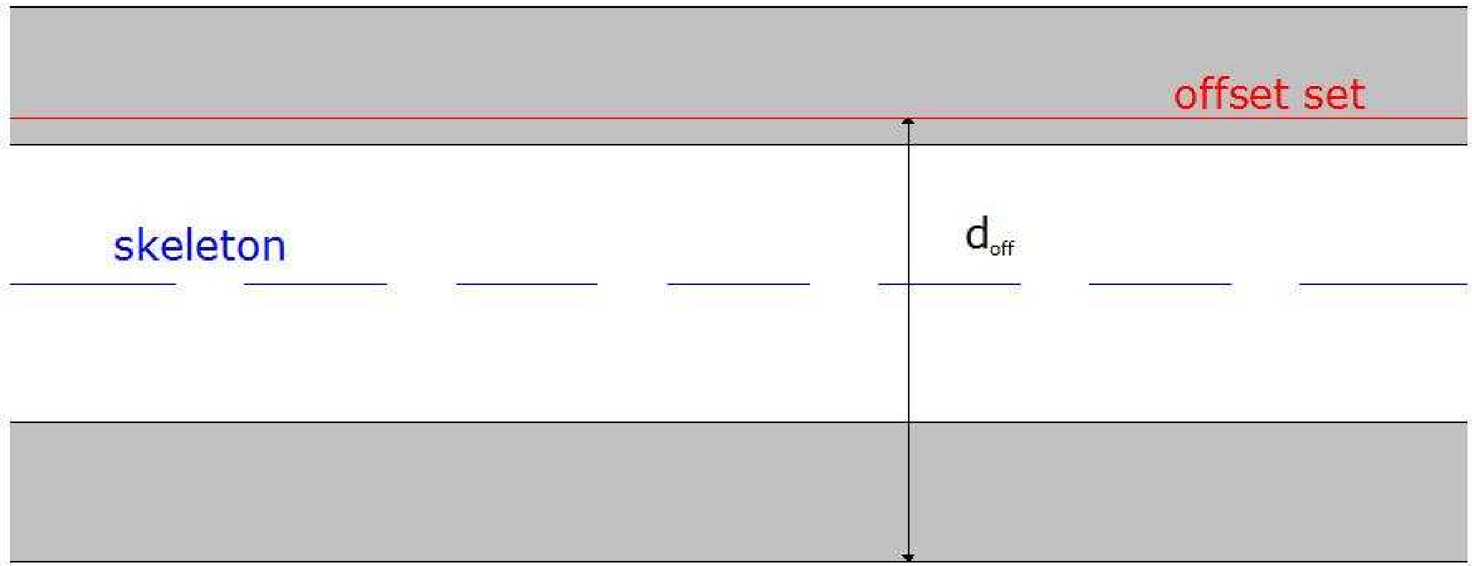


For minimum thicknes we rely on the classical notion of **offset sets** of the boundary of a shape, defined by

$$\{x - d_{\text{off}}\, n(x) \quad \text{such that } x \in \partial\Omega\}$$

Caution with minimum thickness !

Writing a constraint for a single (large) value of $d_{\rm off}$ does not work !



This is the reason why all values of $d_{\rm off}$ between 0 and $d_{\rm min}$ are taken into account.

Quadratic penalty method

We reformulate the pointwise constraint into a global one denoted by $P(\Omega)$.

Maximum thickness

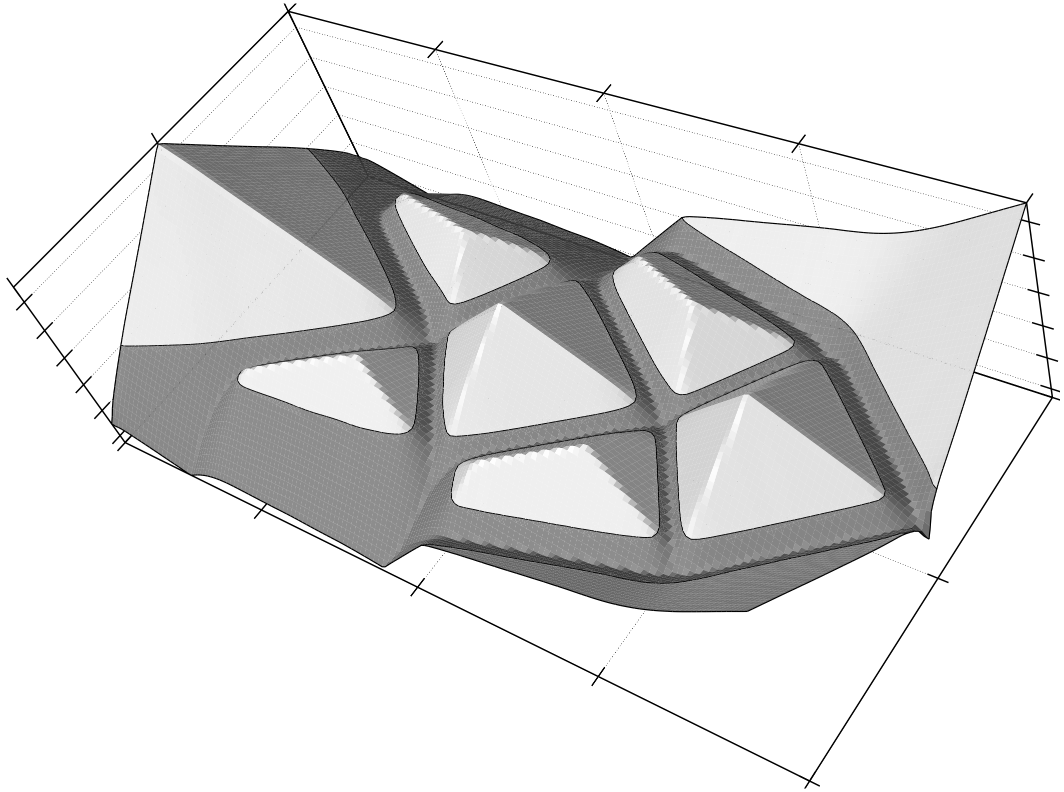
$$P(\Omega) = \int_{\Omega} \left[(d_\Omega(x) + d_{\max}/2)^- \right]^2 dx$$

Minimum thickness

$$P(\Omega) = \int_{\partial\Omega} \int_0^{d_{\min}} \left[(d_\Omega(x - d_{\mathrm{off}} n(x)))^+ \right]^2 dx \, dd_{\mathrm{off}}$$

where $f^+ = \max(f, 0)$ and $f^- = \min(f, 0)$.

Property of the signed distance function



The signed distance function has a tent-like shape.

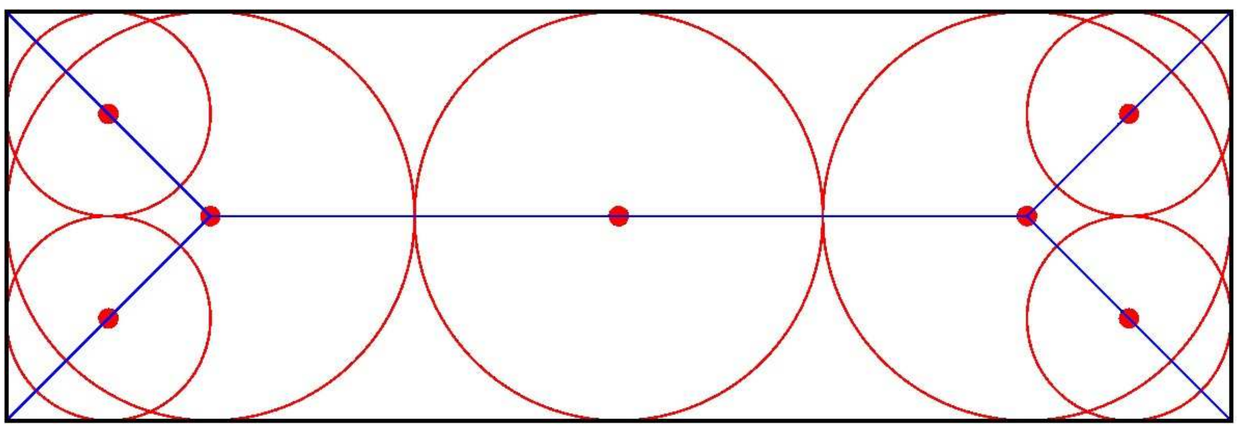
Rays and skeleton of Ω

- The **skeleton** (or ridge) is made of the points $x \in \Omega$ where there are multiple minimizers for

$$d(x, \partial\Omega) = \min_{y \in \partial\Omega} \|x - y\|.$$

- Equivalently, the skeleton is the set of points where $n = \nabla d_\Omega$ is discontinuous.

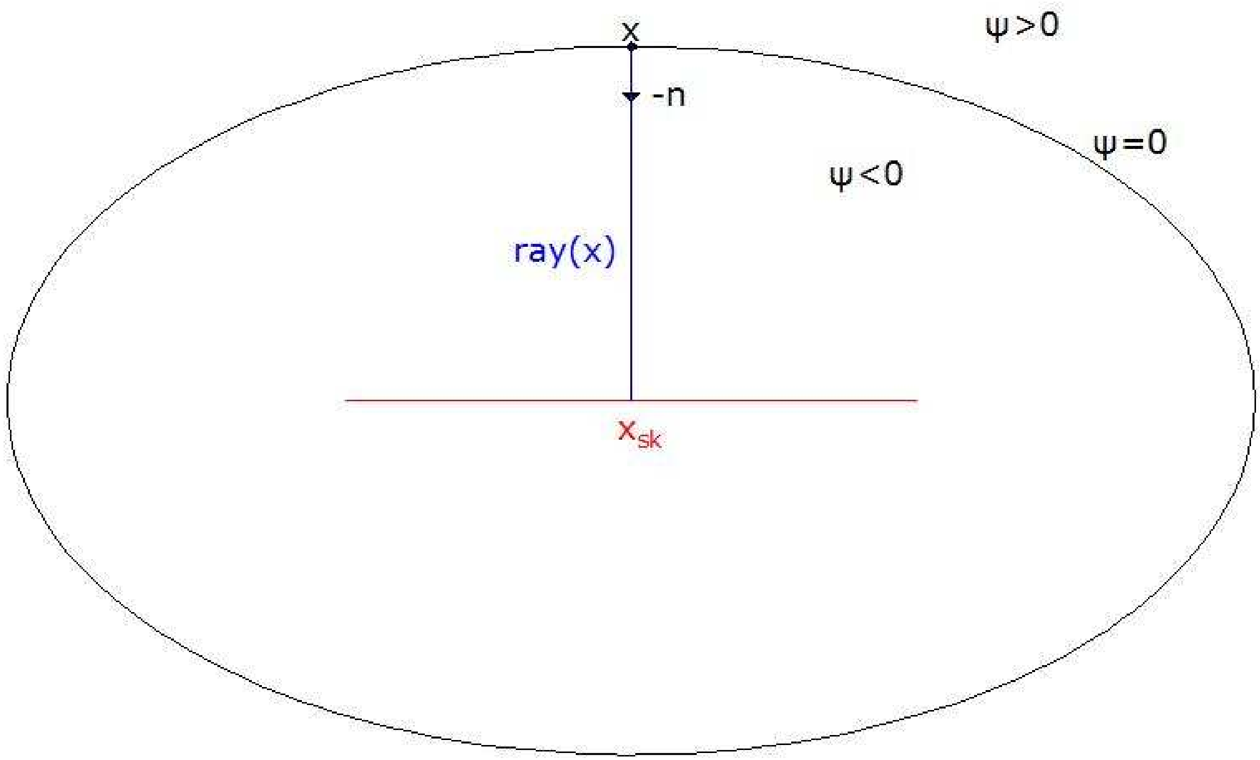
- Equivalently (Huygens principle) the skeleton is the geometric location of centers of maximal disks.



Rays and skeleton (Ctd.)

The **ray** issued from $x \in \partial\Omega$ is the integral curve of $n = \nabla d_\Omega$.

The rays are straight lines because $\dot{x}(t) = n(x(t))$ implies $t = d_\Omega(x(t))$.



Shape derivative of the signed-distance function

Lemma. Fix $x \in \Omega \setminus$ Skeleton. Define $p_{\partial\Omega}(x)$ the unique point on $\partial\Omega$ such that

$$d(x, \partial\Omega) = \|x - p_{\partial\Omega}(x)\|.$$

Then, the "pointwise" shape derivative is

$$d'_\Omega(\theta)(x) = \Big(\theta \cdot n\Big) \left(p_{\partial\Omega}(x)\right).$$

Remarks.

- The computation of the shape derivative of the signed-distance function is classical (e.g. Delfour and Zolesio).
- The shape derivative $d'_\Omega(\theta)$ remains constant along the normal and rays (but is discontinuous on the skeleton).

Lemma. For a function $j \in C^1(\mathbb{R})$ define

$$J(\Omega) = \int_D j(d_\Omega(x))\, dx.$$

Then J is shape differentiable and

$$J'(\Omega)(\theta) = -\int_D j'(d_\Omega(x)) \Big(\theta \cdot n\Big) \left(p_{\partial\Omega}(x)\right)\, dx$$

or equivalently for a C^2 domain Ω (by using a coarea formula)

$$J'(\Omega)(\theta) = -\int_{\partial\Omega} \Big(\theta \cdot n\Big)(y) \left(\int_{\mathrm{ray}(y)} j'(d_\Omega(x)) \prod_{i=1}^{d-1} \left(1 + d_\Omega(x)\kappa_i(y)\right)\, ds \right) dy$$

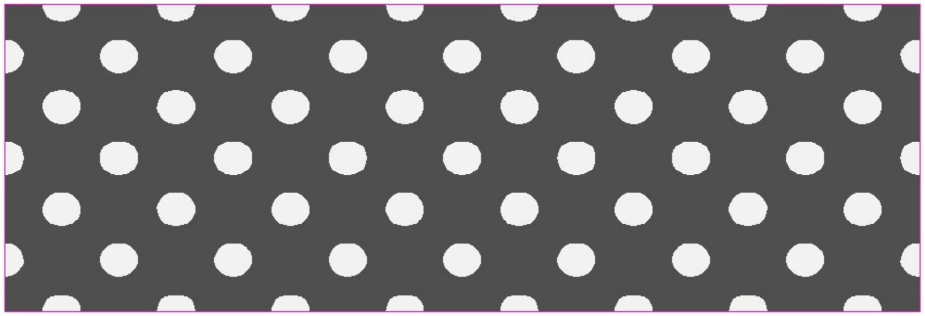
with κ_i the principal curvatures of $\partial\Omega$, $\mathrm{ray}(y) = \{x = y - s\, n(y)\}$ and s the curvilinear abcissa.

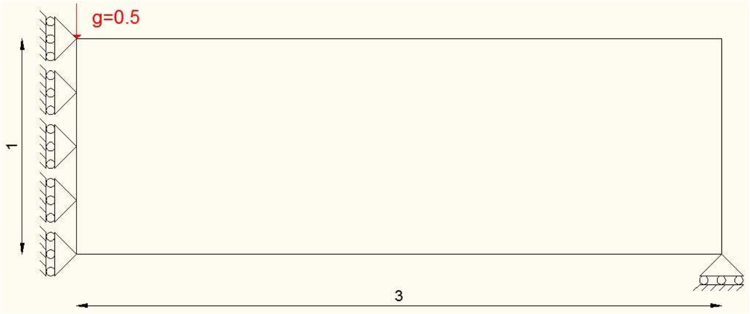
In numerical practice we approximate the Jacobian by 1.

-IV- NUMERICAL RESULTS

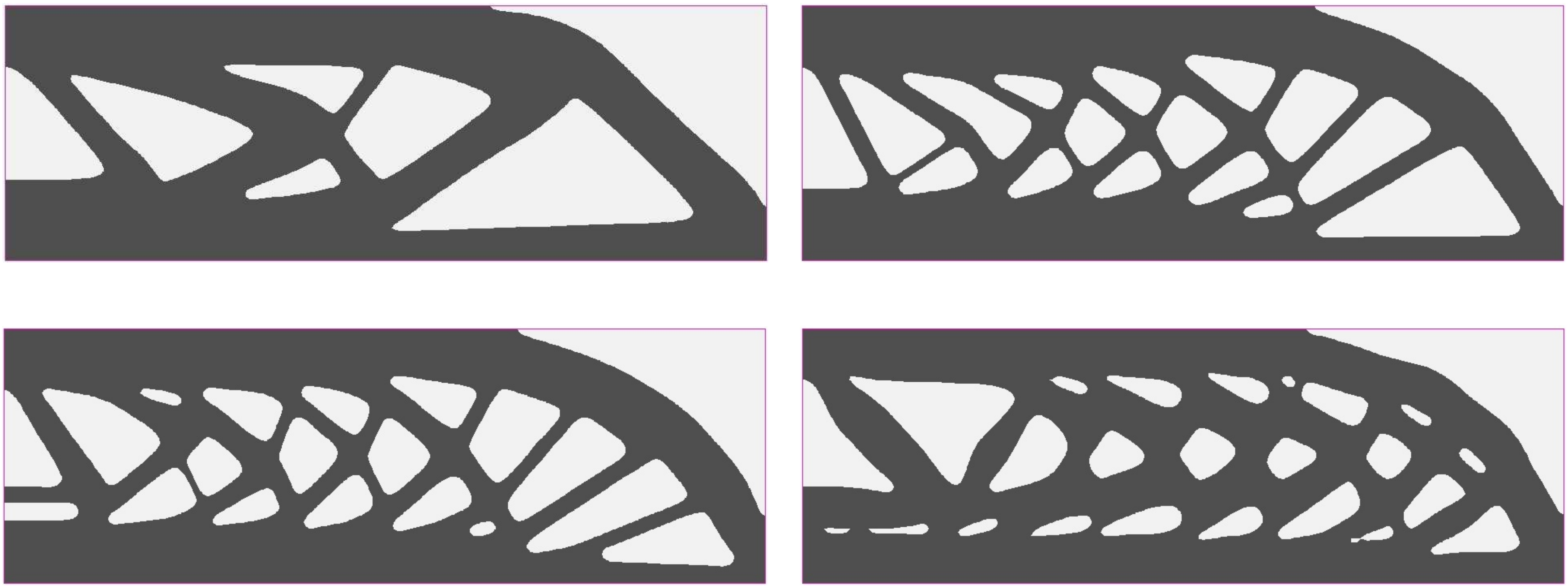
- All the geometrical computations (skeleton, offset, projection, etc.) are standard and very cheap (compared to the elasticity analysis).
- All our numerical examples are for compliance minimization (except otherwise mentioned).
- Optimization: we use an augmented Lagrangian method.
- At convergence, the geometrical constraints are exactly satisfied.
- All results have bee obtained with our software developped in the finite element code SYSTUS of ESI group.

Maximum thickness (MBB, solution without constraint)

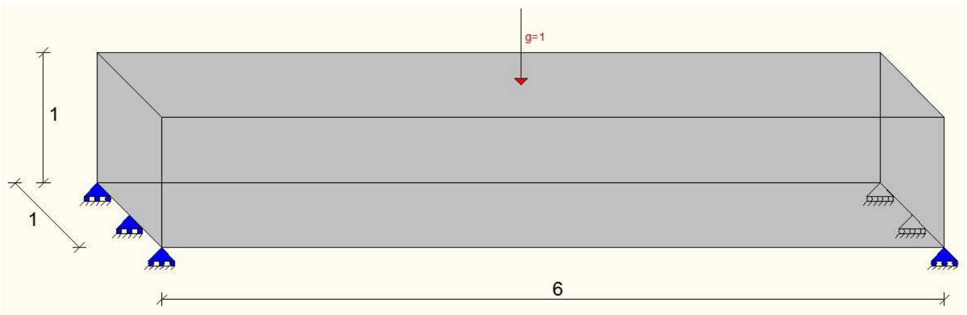


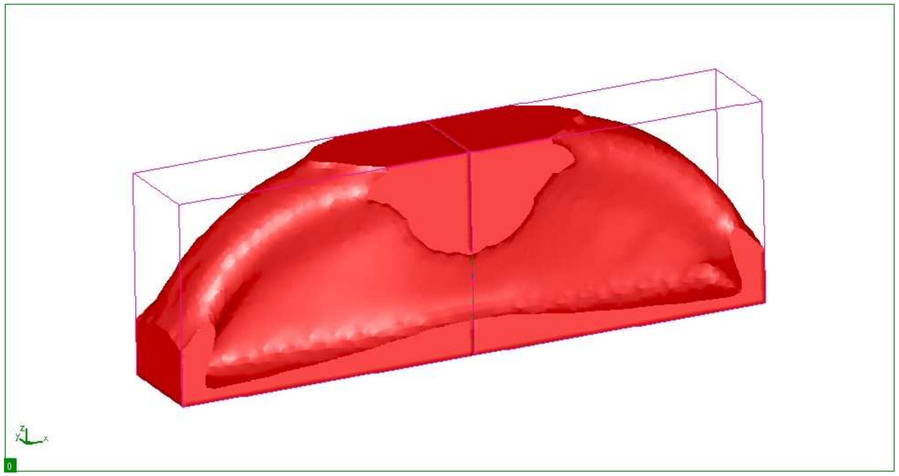


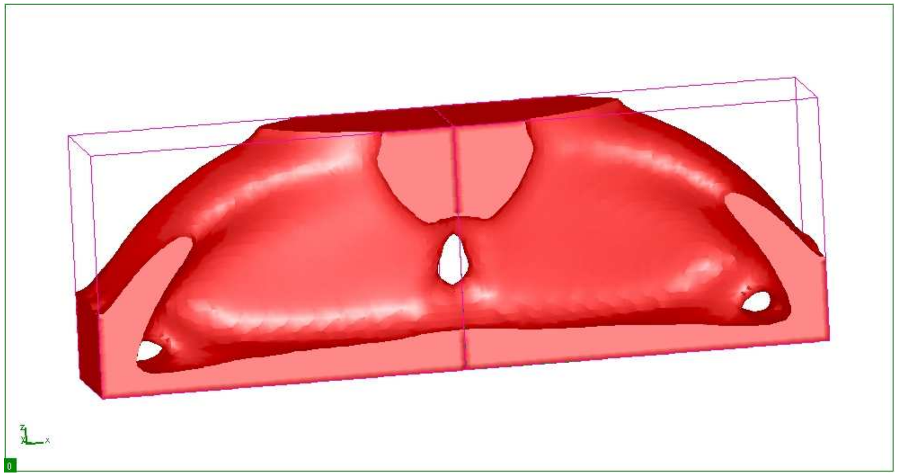
Maximum thickness (solution with increasing constraint)

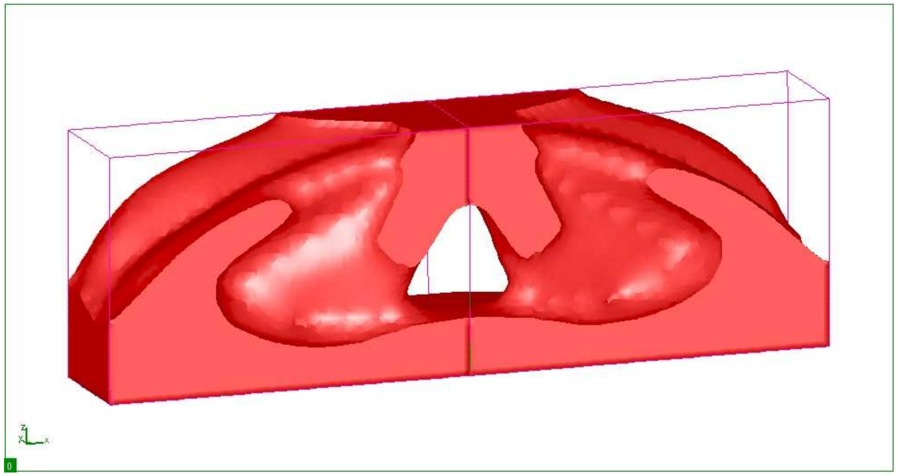


Maximum thickness (3d MBB beam)

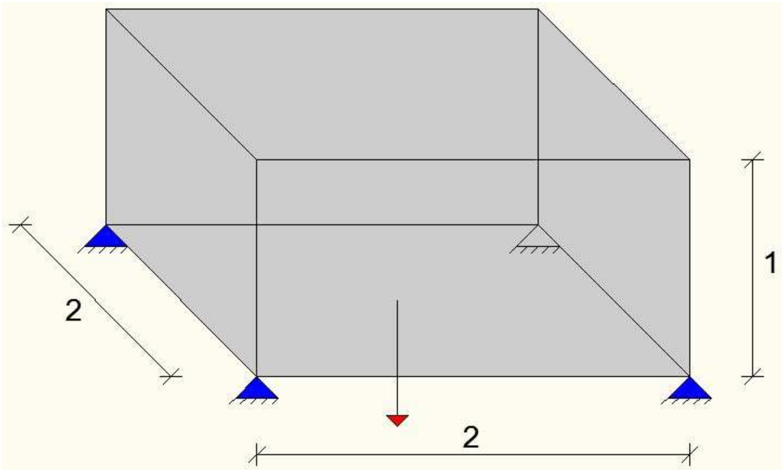


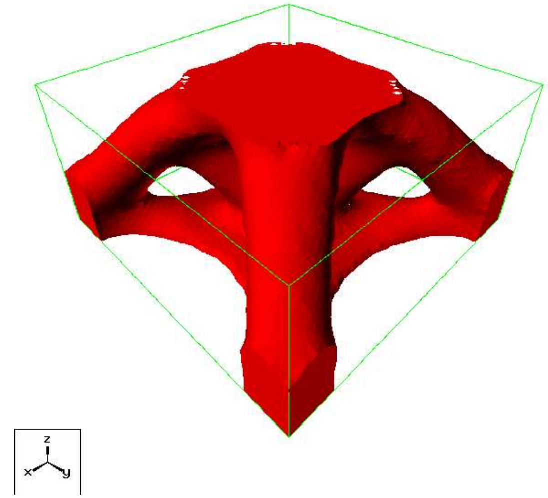


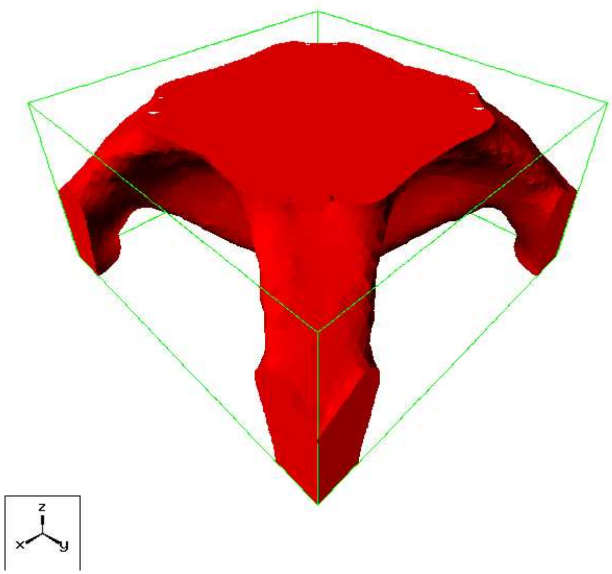


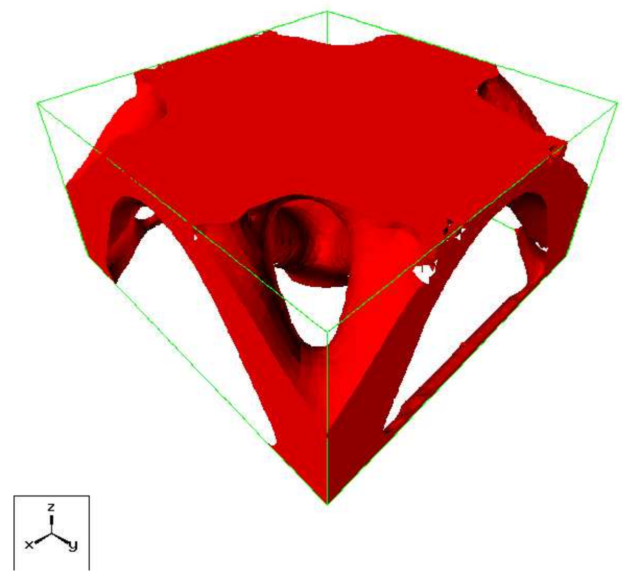


Maximum thickness (3d Box)

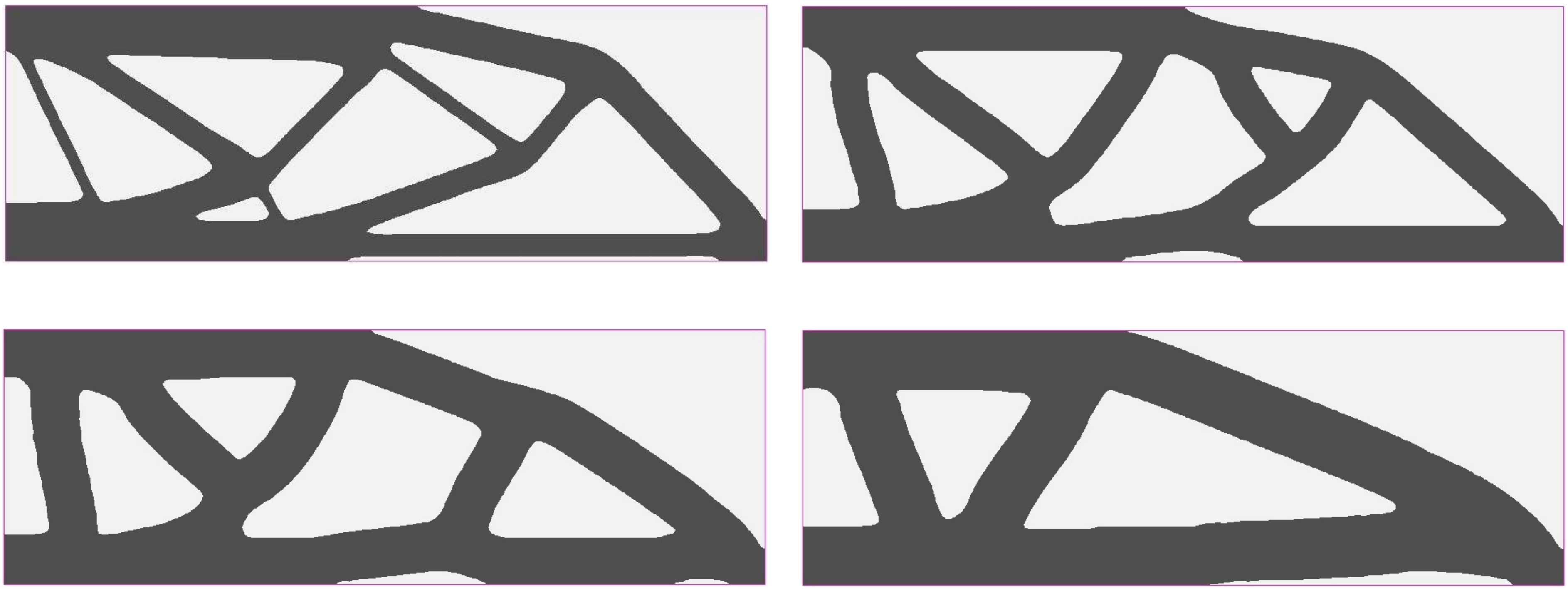




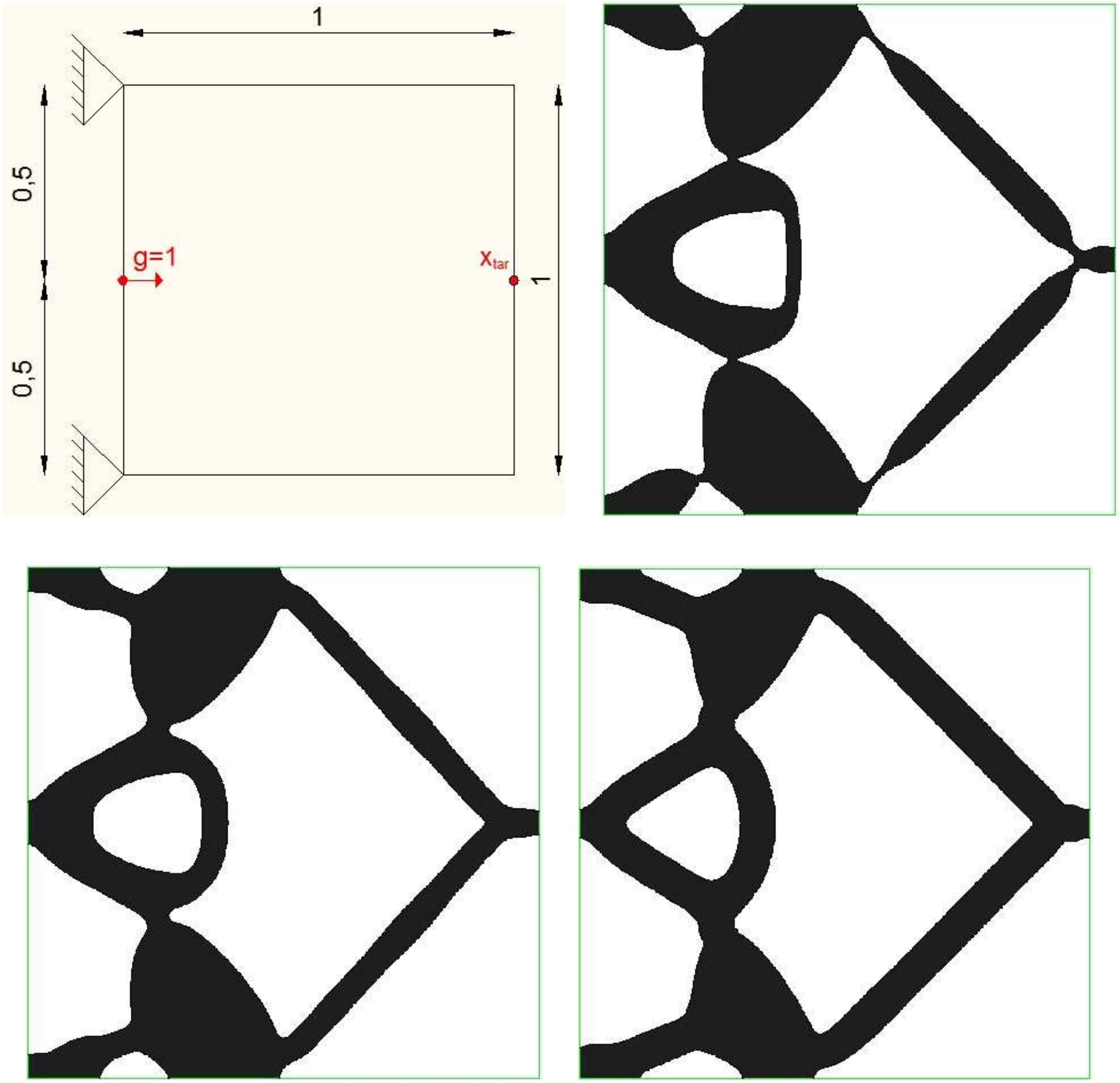




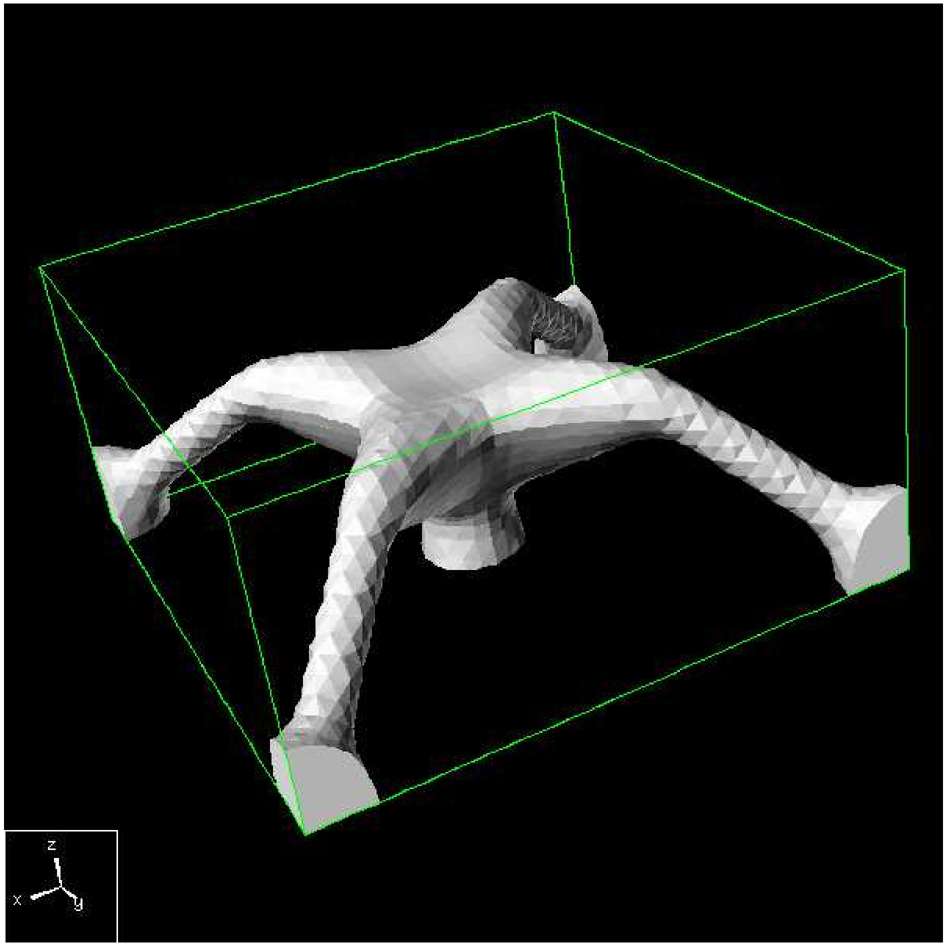
Minimum thickness (MBB beam)

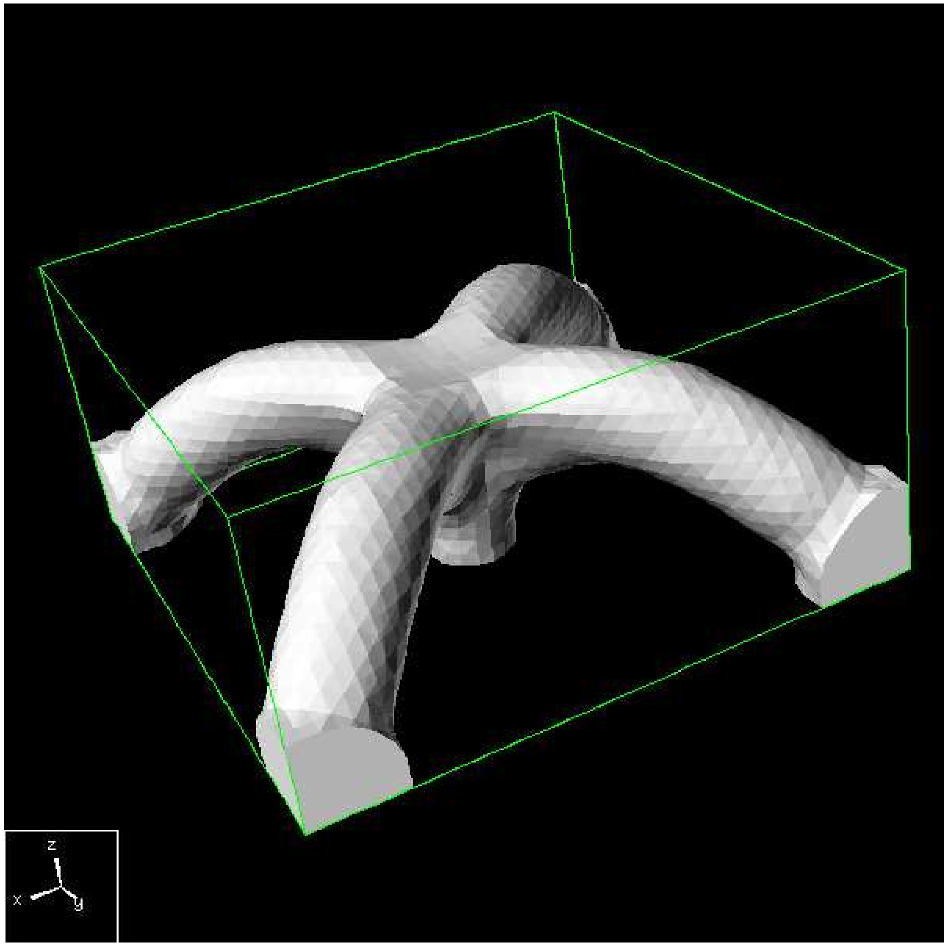


Minimum thickness (force inverter)



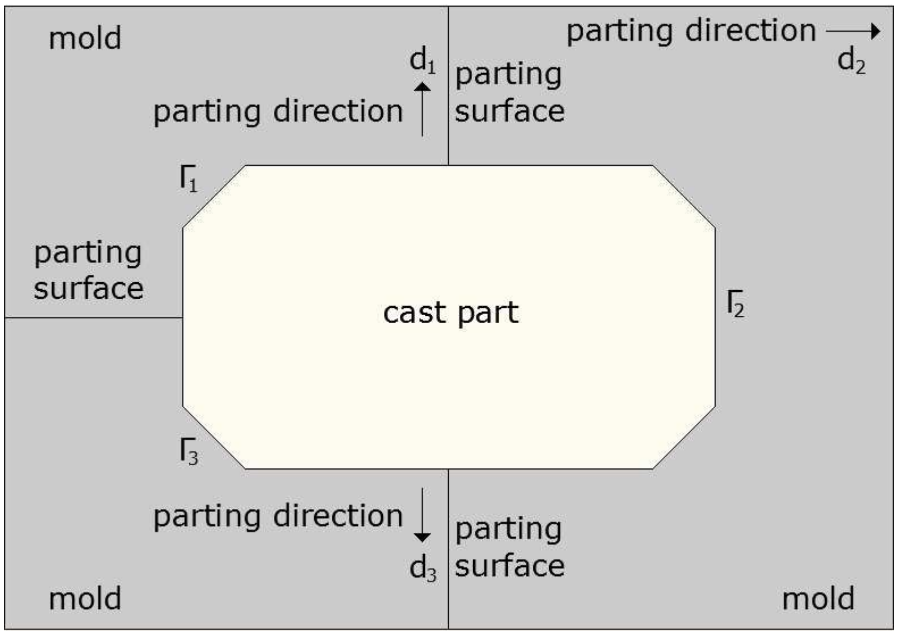
Minimum thickness (3d)

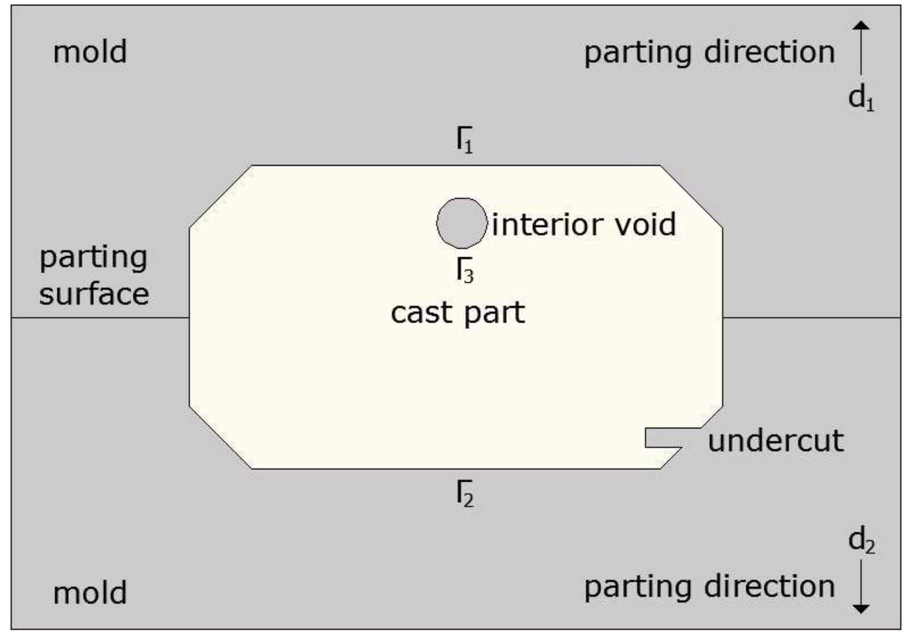




-V- MOLDING CONSTRAINTS

Parting surfaces Γ_i and draw directions d_i: castable (left), not castable (right).





$$\text{Moldability condition:} \quad d_i \cdot n(x) \geq 0, \quad \forall x \in \Gamma_i.$$

Sufficient conditions for molding

Starting from a **castable** initial design:

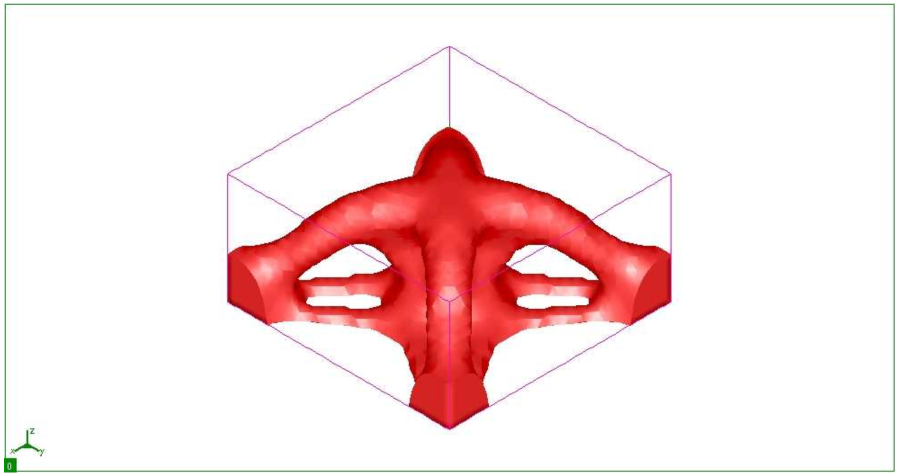
Xia et al. (SMO 2010) proposed to project the velocity

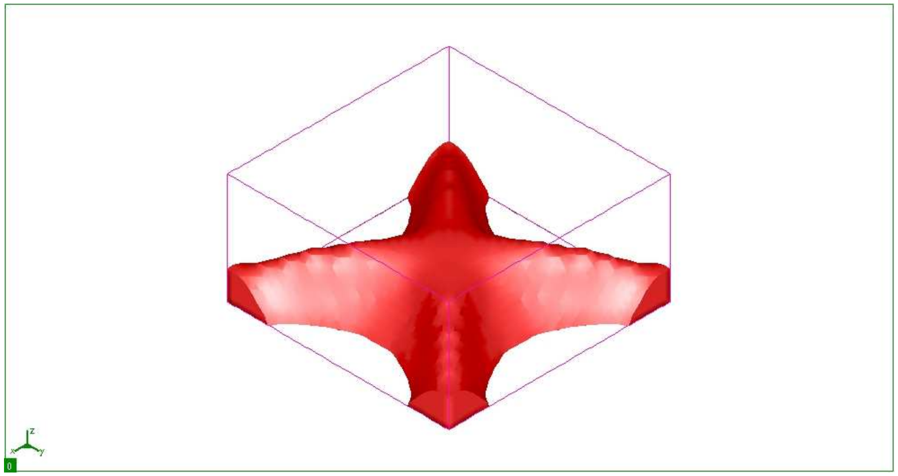
$$\theta_i(x) = \lambda(x) d_i, \quad \forall x \in \Gamma_i.$$

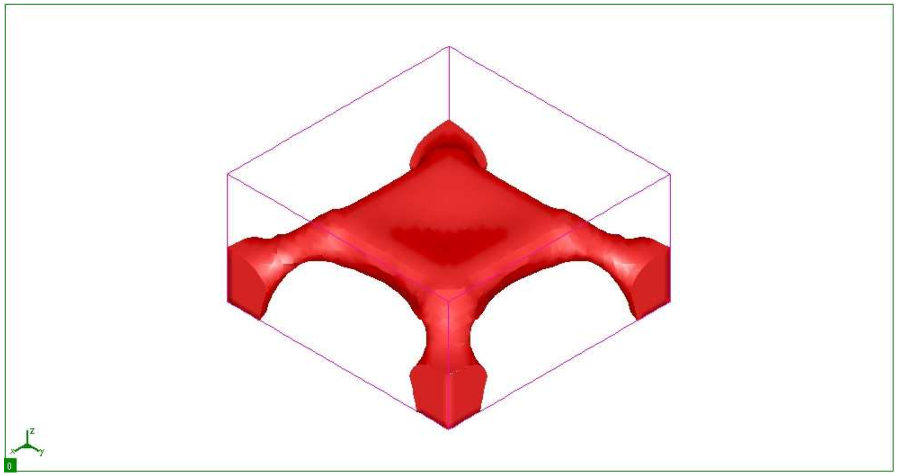
Starting from a **non-castable** initial design:

we suggest the constraint

$$d_\Omega\left(x + \xi d_i\right) \geq 0 \quad \forall x \in \Gamma_i, \quad \forall \xi \in \left[0, dist(x, \partial D)\right].$$

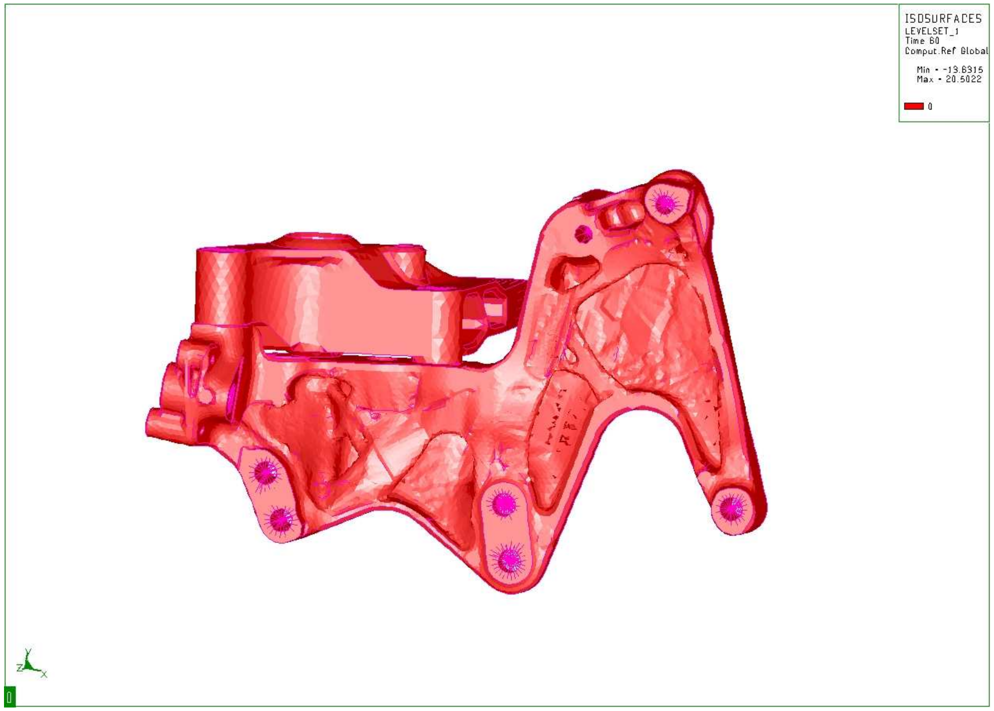


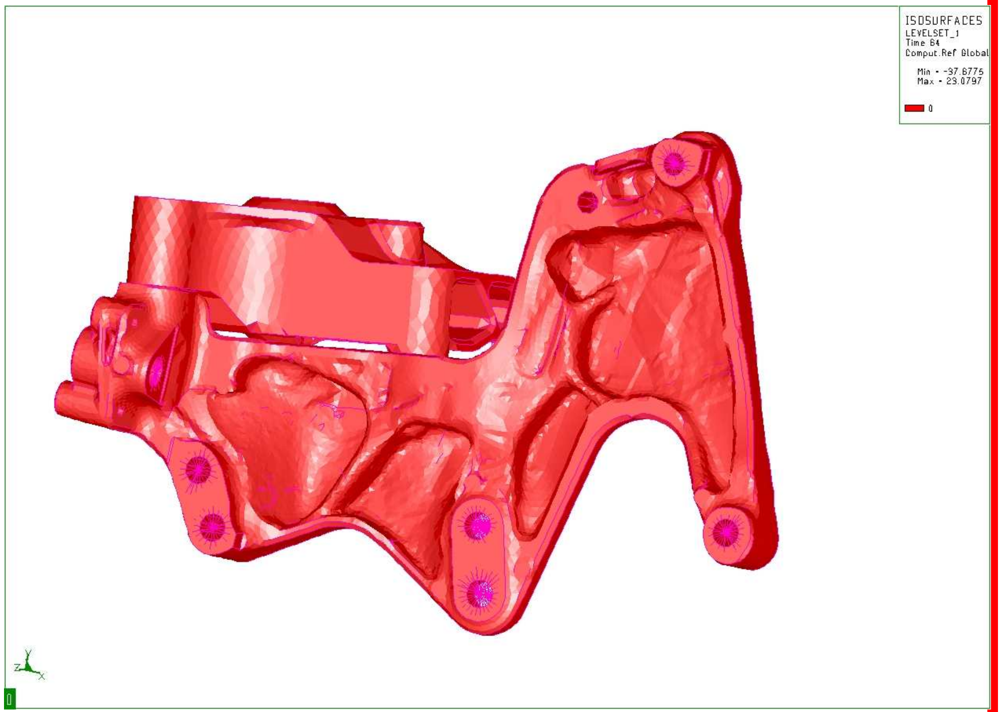




No constraint (top), vertical draw direction (bottom).

Parting surface fixed at bottom (left) and free (right).





Industrial test case (courtesy of Renault): no molding constraint (left), out of plane draw direction (right).

Conclusion

☞ Work still going on.

☞ Other penalizations of the geometrical constraints.

☞ Should we apply the constraints from the start or near the end ?

☞ What if we want to stay feasible at each iteration ?

☞ Handling several constraints simultaneously.

☞ Better optimization algorithm: sequential linear programming with trust region.