PROMYS Europe 2018

Application Problem Set

https://www.promys-europe.org/

Please attempt each of the following problems. Though they can all be solved with no more than a standard high school mathematics background, most of the problems require considerably more ingenuity than is usually expected in high school. You should keep in mind that we do not expect you to find complete solutions to all of them. Rather, we are looking to see how you approach challenging problems.

Here are a few suggestions:

- Think carefully about the meaning of each problem.
- Examine special cases, either through numerical examples or by drawing pictures.
- Be bold in making conjectures.
- Test your conjectures through further experimentation, and try to devise mathematical proofs to support the surviving ones.
- Can you solve special cases of a problem, or state and solve simpler but related problems?

If you think you know the answer to a question, but cannot prove that your answer is correct, tell us what kind of evidence you have found to support your belief. If you use books, articles or websites in your explorations, be sure to cite your sources.

Be careful about how you search online for help. We are interested in your ideas, not in solutions that you have found elsewhere. If you search online for a problem and find a solution (or most of a solution), it will be much harder for you to demonstrate your insight to us.

You may find that most of the problems require some patience. Do not rush through them. It is not unreasonable to spend a month or more thinking about the problems. It might be good strategy to devote most of your time to a small selection of problems which you find especially interesting. Be sure to tell us about progress you have made on problems not yet completely solved. **For each problem you solve, please justify your answer clearly and tell us how you arrived at your solution.**

1. Calculate each of the following:

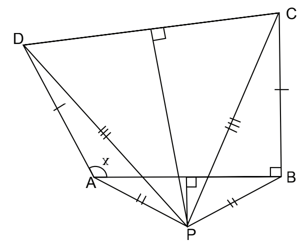
$$\begin{aligned} 1^3 + 5^3 + 3^3 &= ?? \\ 16^3 + 50^3 + 33^3 &= ?? \\ 166^3 + 500^3 + 333^3 &= ?? \\ 1666^3 + 5000^3 + 3333^3 &= ?? \end{aligned}$$

What do you see? Can you state and prove a generalization of your observations?

2. The sequence (x_n) of positive real numbers satisfies the relationship $x_{n-1}x_nx_{n+1} = 1$ for all $n \geq 2$. If $x_1 = 1$ and $x_2 = 2$, what are the values of the next few terms? What can you say about the sequence? What happens for other starting values?

The sequence (y_n) satisfies the relationship $y_{n-1}y_{n+1} + y_n = 1$ for all $n \geq 2$. If $y_1 = 1$ and $y_2 = 2$, what are the values of the next few terms? What can you say about the sequence? What happens for other starting values?

3. According to the Journal of Irreproducible Results, any obtuse angle is a right angle!



Here is their argument. Given the obtuse angle x, we make a quadrilateral $ABCD$ with $\angle DAB = x$, and $\angle ABC = 90°$, and $AD = BC$. Say the perpendicular bisector to DC meets the perpendicular bisector to AB at P. Then $PA = PB$ and $PC = PD$. So the triangles PAD and PBC have equal sides and are congruent. Thus $\angle PAD = \angle PBC$. But PAB is isosceles, hence $\angle PAB = \angle PBA$. Subtracting, gives $x = \angle PAD - \angle PAB = \angle PBC - \angle PBA = 90°$. This is a preposterous conclusion – just where is the mistake in the "proof" and why does the argument break down there?

4. The set S contains some real numbers, according to the following three rules.

(i) $\frac{1}{1}$ is in S.

(ii) If $\frac{a}{b}$ is in S, where $\frac{a}{b}$ is written in lowest terms (that is, a and b have highest common factor 1), then $\frac{b}{2a}$ is in S.

(iii) If $\frac{a}{b}$ and $\frac{c}{d}$ are in S, where they are written in lowest terms, then $\frac{a+c}{b+d}$ is in S.

These rules are exhaustive: if these rules do not imply that a number is in S, then that number is not in S. Can you describe which numbers are in S?

5. We say that a positive integer is *quiteprime* if it is not divisible by 2, 3, or 5. How many quiteprime positive integers are there less than 100? less than 1000? A positive integer is *very quiteprime* if it is not divisible by any prime less than 15. How many very quiteprime positive integers are there less than 90000? Without giving an exact answer, can you say *approximately* how many very quiteprime positive integers are less than 10^{10}? less than 10^{100}? Explain your reasoning as carefully as you can.

6. To get the *copycopy* of a positive integer, we write it twice without a space. For example, the copycopy of 2018 is 20182018. Is there a positive integer whose copycopy is a perfect square? If so, how many such positive integers can you find?

7. A unit fraction is a fraction of the form $\frac{1}{n}$ where n is a positive integer. Note that the unit fraction $\frac{1}{11}$ can be written as the sum of two unit fractions in the following three ways:

$$\frac{1}{11} = \frac{1}{12} + \frac{1}{132} = \frac{1}{22} + \frac{1}{22} = \frac{1}{132} + \frac{1}{12}.$$

Are there any other ways of decomposing $\frac{1}{11}$ into the sum of two unit fractions? In how many ways can we write $\frac{1}{60}$ as the sum of two unit fractions? More generally, in how many ways can the unit fraction $\frac{1}{n}$ be written as the sum of two unit fractions? In other words, how many ordered pairs (a, b) of positive integers a, b are there for which

$$\frac{1}{n} = \frac{1}{a} + \frac{1}{b} \ ?$$

8. Let P_0 be an equilateral triangle of area 10. Each side of P_0 is trisected, and the corners are snipped off, creating a new polygon (in fact, a hexagon) P_1. What is the area of P_1? Now repeat the process to P_1 – i.e. trisect each side and snip off the corners – to obtain a new polygon P_2. What is the area of P_2? Now repeat this process infinitely often to create an object P_∞. What is the area of P_∞?

9. Let S be a set of positive real numbers. If S contains at least four distinct elements show that there are elements $x, y \in S$ such that

$$0 < \frac{x - y}{1 + xy} < \sqrt{3}/3.$$

What can you say if S has at least 7 elements? What if S has at least n elements, where $n > 2$?

10. A giant rabbit is tied to a pole in the ground by an infinitely stretchy elastic cord attached to its tail. A hungry flea is on the pole watching the rabbit. The rabbit sees the flea, jumps into the air and lands one kilometre from the pole (with its tail still attached to the pole by the elastic cord). The flea gives chase and leaps into the air landing on the stretched elastic cord one centimetre from the pole. The rabbit, seeing this, again leaps into the air and lands another kilometre away from the pole (i.e., a total of two kilometres from the pole). Undaunted, the flea bravely leaps into the air again, landing on the elastic cord one centimetre further along. Once again the rabbit jumps another kilometre and the flea jumps another centimetre along the cord. If this continues indefinitely, will the flea ever catch up to the rabbit? (Assume the earth is flat and extends infinitely far in all directions.)