Complex Geometry and Gravitational Instantons

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From Good Cuts to Celestial Holography

Oxford, July 2025

Joint work with Lars Andersson

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The Chen-Teo instanton

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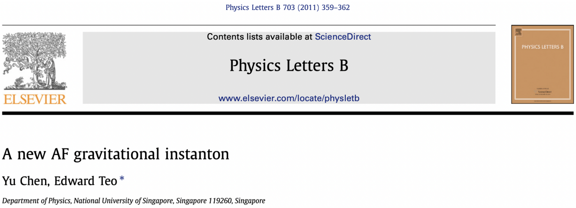
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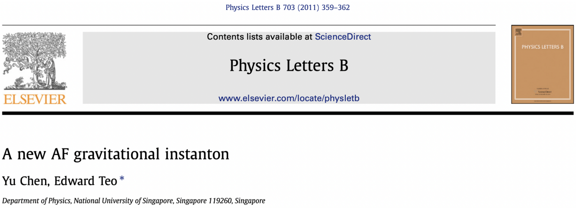

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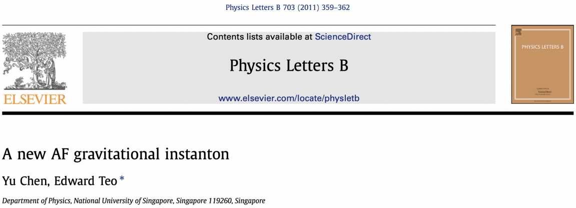

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- Need to have better understanding of Moduli Spaces

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For general M:

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- $\mathscr{E}(M)$ may be a non-smooth variety
- $\mathscr{E}(M)$ is **integrable** at g_0 if any infinitesimal deformation integrates to a curve of Einstein metrics in $\mathscr{E}(M)$

Integrability

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- Calabi-Yau (cf. Besse 1987); self-dual Yang-Mills (Atiyah, Hitchin, Singer 1978); twistor theory & deformations of complex structures (Penrose, Kodaira) ✓

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- Is the Moduli Space of ALF instantons integrable?
- We will be able to answer this using complex structures

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Theorem B (Integrability) [B.A & L. Andersson, 2025]

Let (M, g) be an ALF Hermitian non-Kähler gravitational instanton. Then:

- *Infinitesimal deformations are conformally Kähler.*
- *The Moduli Space $\mathscr{E}(M)$ is integrable at g.*

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- *The Moduli Space $\mathscr{E}(M)$ is integrable at g.*

We will focus on **Theorem B**, using "Witten-like" identity (in the sense of PMT)

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Remark: A global spin structure is not assumed.

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- $\hat{\varphi}_{AB}$ with $\hat{\Psi}_{AB}{}^{CD}\hat{\varphi}_{CD} = \hat{\lambda}_3\hat{\varphi}_{AB}$ and $|\hat{\varphi}|^2 = 1$

Key identity

Lemma

With the above definitions, let $(M, \hat{g}_{ab})$ be arbitrary $(\hat{\lambda}_3 \neq 0)$, and set

$$\hat{S}_B^{A'} = \hat{\nabla}^{AA'} \hat{\varphi}_{AB}, \qquad \hat{V}^a = \hat{\varphi}^{AB} \hat{S}_B^{A'}, \qquad \hat{P}^{abcd} = \hat{\varphi}^{(AB} \hat{\varphi}^{CD)} \hat{\epsilon}^{A'B'} \hat{\epsilon}^{C'D'}$$

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$$\mathcal{B} = -\tfrac{1}{3}\hat{P}^{abcd}\hat{\nabla}_a(\hat{\lambda}_3^{-1}\nabla_d E_{bc})$$

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Remark

This result was obtained by Wu (2019) and LeBrun (2019) in the case that (M, g) is compact, Einstein and $\det W^+ > 0$

ALF curves

Recall key identity $\hat{\nabla}_a \hat{V}^a = \mathcal{A} + \mathcal{B}$.

Lemma

Let $g(s)$ be a curve of ALF metrics on M, with $g_0 := g(0)$ Hermitian instanton and $\delta g := \frac{\mathrm{d}g}{\mathrm{d}s}|_{s=0}$ infinitesimal deformation. Then

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Sketch of proof

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- Using that g_0 is ALF Hermitian instanton,

$$\mathcal{A} = \left\{ \frac{|\delta(\hat{\nabla}\hat{J})|^2}{24} + |\delta\hat{S}|^2 + \frac{(\delta\hat{\lambda}_1 - \delta\hat{\lambda}_2)^2}{3\hat{\lambda}_3} + \frac{(|\delta\hat{\Gamma}^3{}_1|^2 + |\delta\hat{\Gamma}^3{}_2|^2)}{3} \right\} \frac{s^2}{2!} + \mathcal{O}(s^3)$$

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- Hermitian moduli spaces are smooth $\Rightarrow$ δg integrates to a curve of Ricci-flat metrics in $\mathscr{E}(M)$.

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Thank you!