Advances in Flat Holography

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Abstract: These lectures will discuss applications of twistor theory to celestial holography. We will review topological strings and branes and use them to construct holographic duals of certain local holomorphic theories in twistor space. By the Penrose transform, local holomorphic theories on twistor space give rise to integrable theories on spacetime. Their celestial duals arise as 2d chiral CFTs living on D-branes. With pedagogical examples, we will show that 2d CFT correlators reproduce scattering amplitudes in the bulk.

Contents

1 Lecture 1 **1**

2 Lecture 2 **4**

3 Lecture 3 **7**

1 Lecture 1

➤ **Big picture:** Celestial holography = twisted holography on twistor space.
Allows for a Maldacena-style derivation of simple examples of celestial holography.

➤ **Goals of these lectures:**

- Basics of B-model branes
- Celestial holography via twistor space
- Soft algebras + amplitudes

➤ **Main references:**

- arXiv: 2306.00940 (Burns space and holography)
- arXiv: 2412.02680 (Self-Dual Gauge Theory from the Top Down)

➤ σ-model on a Calabi-Yau target with complex coordinates $x^i, \bar{x}^{\bar{\imath}}$ and metric $g_{i\bar{\jmath}}$,

$$S = \int_\Sigma \mathrm{d}^2 z \left\{ \frac{g_{i\bar{\jmath}}}{2} (\partial x^i \bar{\partial} \bar{x}^{\bar{\jmath}} + \bar{\partial} x^i \partial \bar{x}^{\bar{\jmath}}) + g_{i\bar{\jmath}} \, \psi^i \bar{\partial} \psi^{\bar{\jmath}} + g_{i\bar{\jmath}} \, \tilde{\psi}^i \partial \tilde{\psi}^{\bar{\jmath}} - R_{i\bar{\jmath}k\bar{l}} \, \psi^i \psi^{\bar{\jmath}} \tilde{\psi}^k \tilde{\psi}^{\bar{l}} \right\} , \quad (1.1)$$

where ψ^i has weights $(1,0)$, $\tilde{\psi}^i$ has weights $(0,1)$, and $\psi^{\bar{\imath}}, \tilde{\psi}^{\bar{\imath}}$ have weights $(0,0)$.

➤ Perform the field redefinitions

$$\eta^{\bar{\imath}} = \psi^{\bar{\imath}} + \tilde{\psi}^{\bar{\imath}} , \qquad \theta_i = g_{i\bar{\jmath}} (\psi^{\bar{\jmath}} - \tilde{\psi}^{\bar{\jmath}}) . \quad (1.2)$$

In terms of these, the σ-model has a nilpotent supersymmetry,

$$\begin{aligned} &Q x^i = 0 , \qquad Q \bar{x}^{\bar{\imath}} = \eta^{\bar{\imath}} , \qquad Q \eta^{\bar{\imath}} = Q \theta_i = 0 , \\ &Q \psi^i = \partial x^i , \qquad Q \tilde{\psi}^i = \bar{\partial} x^i \end{aligned} \quad (1.3)$$

Clearly $Q^2 = 0$.

➤ B-model is the twisted subsector obtained by taking Q-cohomology. This is a topological subsector because the worldsheet stress tensor becomes Q-exact, whereby correlators of Q-closed operators $\mathcal{O}(z_i, \bar{z}_i)$ do not depend on their insertion points $(z_i, \bar{z}_i)$.

- We want to study
  - Closed string spectrum: BCOV theory
  - Open string spectrum: holomorphic Chern-Simons theory
  - Orientifold projection (next lecture)
- Let us start with the closed string spectrum. There is a worldsheet-target correspondence
$$\eta^{\bar{\imath}} \;\leftrightarrow\; \mathrm{d}\bar{x}^{\bar{\imath}}\,, \qquad \theta_i \;\leftrightarrow\; \partial_i\,. \tag{1.4}$$
To every target space field
$$\Phi = \Phi^{k\cdots l}_{\bar{\imath}\cdots\bar{\jmath}}(x,\bar{x})\,\mathrm{d}\bar{x}^{\bar{\imath}}\wedge\cdots\wedge\mathrm{d}\bar{x}^{\bar{\jmath}}\,\partial_k\wedge\cdots\wedge\partial_l\,, \tag{1.5}$$
we can associate a $(h,\bar{h})=(0,0)$ worldsheet primary
$$V[\Phi] \;= \Phi^{k\cdots l}_{\bar{\imath}\cdots\bar{\jmath}}(x,\bar{x})\,\eta^{\bar{\imath}}\cdots\eta^{\bar{\jmath}}\,\theta_k\cdots\theta_l\,. \tag{1.6}$$
- The "BRST charge" $Q=\eta^{\bar{\imath}}\partial_{\bar{\imath}}+\cdots$ acts as a worldsheet pullback of $\bar{\partial}=\mathrm{d}\bar{x}^{\bar{\imath}}\,\partial_{\bar{\imath}}$,
$$QV[\Phi] = V[\bar{\partial}\Phi]\,. \tag{1.7}$$
So worldsheet Q-cohomology gets identified with target space Dolbeault cohomology.
- Physically, the most interesting closed string mode is the *Beltrami differential*
$$\Phi = \beta = \beta^i_{\bar{\jmath}}\,\mathrm{d}\bar{x}^{\bar{\jmath}}\,\partial_i\,. \tag{1.8}$$
This governs gravity at the level of complex structure deformations $\bar{\partial}\mapsto\bar{\partial}+\beta$.
- Nonlinearly, it is found to obey the field equation (in absence of other fields)
$$\bar{\partial}\beta+\frac{1}{2}\,[\beta,\beta]=0 \tag{1.9}$$
where $[-,-]$ denotes the Lie bracket of vector fields. It is also required to obey an off-shell constraint of being divergence-free (which arises from level-matching on the worldsheet):
$$\partial(\beta\lrcorner\Omega)=0\,, \tag{1.10}$$
where Ω is the Calabi-Yau volume form and $\partial=\mathrm{d}x^i\,\partial_i$. Similar field equations and constraints hold for other Φ.
- The associated action principle is a non-local action known as *BCOV theory*. Eg., on a Calabi-Yau 3-fold, the action for β takes the form
$$S=\int_{\mathrm{CY}_3}\Omega\wedge\left(\frac{1}{2}\,\partial^{-1}\beta\,\bar{\partial}\beta+\frac{1}{3!}\,\beta^3\right)\lrcorner\,\Omega\,. \tag{1.11}$$
We will not use this too explicitly.

- Next let's study the open string spectrum on Dp-branes for p odd. They necessarily wrap holomorphic submanifolds. The boundary conditions obeyed by fermions are

  – (NN) $\psi^{\bar{\imath}}|_{\partial\Sigma} = \tilde{\psi}^{\bar{\imath}}|_{\partial\Sigma} \implies \theta_i|_{\partial\Sigma} = 0$
  – (DD) $\psi^{\bar{\imath}}|_{\partial\Sigma} = -\tilde{\psi}^{\bar{\imath}}|_{\partial\Sigma} \implies \eta^{\bar{\imath}}|_{\partial\Sigma} = 0$

  So open string vertex operators can contain $x^i, \bar{x}^{\bar{\imath}}, \eta^{\bar{\imath}}$ in directions parallel to the brane and θ_i in directions orthogonal to the brane.

- Strings on space-filling branes can only contain $x^i, \bar{x}^{\bar{\imath}}, \eta^{\bar{\imath}}$ and no θ_i. The general string field is a direct sum of $(0,p)$-forms,

$$\begin{aligned} \mathcal{A} &= c + a_{\bar{\imath}}\, \mathrm{d}\bar{x}^{\bar{\imath}} + \cdots\,, \\ V[\mathcal{A}] &= c + a_{\bar{\imath}}\eta^{\bar{\imath}} + \cdots\,. \end{aligned} \tag{1.12}$$

  The physical field here is the partial connection $a = a_{\bar{\imath}}\, \mathrm{d}\bar{x}^{\bar{\imath}}$, which acts like a gauge field.

- BRST closure gives $QV[\mathcal{A}] = 0 \implies \bar{\partial}\mathcal{A} = 0$. Nonlinearly, it obeys

$$\bar{\partial}\mathcal{A} + \mathcal{A}^2 = 0\,. \tag{1.13}$$

  The associated action principle is known as *holomorphic-Chern-Simons theory*,

$$S = \int_{\mathrm{CY}} \mathrm{Tr}\left(\frac{1}{2}\,\mathcal{A}\,\bar{\partial}\mathcal{A} + \frac{1}{3}\,\mathcal{A}^3\right) \wedge \Omega\,. \tag{1.14}$$

  which works on a Calabi-Yau of any dimension, in particular in complex dimension 3.

- We will study the B-model on a CY 3-fold obtained from twistor space, and will back-react by a stack of N D1 branes. So let us figure out the "celestial dual" theory living on such a stack.

- It suffices to work on flat space $\mathbb{C}^3$ with coordinates x, y, z (not to be confused with the worldsheet coordinate z). Wrap a stack of N D1 branes along the $\mathbb{C}$-plane $x = y = 0$. I.e., impose Neumann boundary conditions along z and Dirichlet boundary conditions along x, y. Then open string vertex operators can depend on $z, \bar{z}, \eta^{\bar{z}} \equiv \mathrm{d}\bar{z}$ and $\theta_x \equiv \partial_x, \theta_y \equiv \partial_y$:

$$\mathcal{A} = c + X\,\partial_x + Y\,\partial_y + b\,\partial_x \wedge \partial_y + \text{fields containing } \mathrm{d}\bar{z} \tag{1.15}$$

  where X, Y are bosonic and b, c are fermionic $\mathfrak{gl}_N$-valued fields on the D1 branes. The fields with $\mathrm{d}\bar{z}$ can be set to zero as a gauge fixing condition.

- Their action is a generalization of the hCS action:

$$S = \int_{\mathrm{D1}} \mathrm{Tr}\left(\frac{1}{2}\,\mathcal{A}\,\bar{\partial}\mathcal{A} + \frac{1}{3}\,\mathcal{A}^3\right) \lrcorner\, \Omega\,, \tag{1.16}$$

  where $\Omega = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$ in this case. The action for the Higgs fields X, Y and ghosts b, c takes the form

$$S = \int_{\mathbb{C}} \mathrm{Tr}\,(X\bar{\partial}Y + b\bar{\partial}c) \wedge \mathrm{d}z\,. \tag{1.17}$$

  This hCS construction works for all branes. This is the kind of celestial dual theories that we will encounter in the coming lectures.

2 Lecture 2

- The previous lecture dealt with branes in the B-model. Today, we will use them to construct holographic dualities. We will work with a type I version of the B-model on twistor space. Our spacetime signature will be Euclidean.

- Recall from Roland's lectures that twistor space is $\mathbb{PT} = \mathscr{O}(1) \oplus \mathscr{O}(1) \to \mathbb{P}^1$. We give $\mathbb{P}^1$ an affine coordinate z, and the fiber coordinates are denoted $v^{\dot\alpha} = (v^{\dot 0}, v^{\dot 1})$. $\mathbb{PT}$ is diffeomorphic (but not biholomorphic) to $\mathbb{R}^4 \times \mathbb{P}^1$ through the map

$$v^{\dot 0} = u^{\dot 0} - z\bar{u}^1\,, \qquad v^{\dot 1} = u^{\dot 1} + z\bar{u}^0\,, \tag{2.1}$$

where $u^{\dot\alpha}$ are complex coordinates on $\mathbb{R}^4 = \mathbb{C}^2$, and $\bar{u}^\alpha$ their complex conjugates. These are Penrose's incidence relations in Euclidean signature.

- We want to study holography for the B-model on $\mathbb{PT}$, and thereby for the associated integrable theories on $\mathbb{R}^4$ obtained from compactification along $\mathbb{P}^1$.

- But sadly $\mathbb{PT}$ is not Calabi-Yau. Coordinates on $\mathscr{O}(1)$ transform under $z \mapsto z^{-1}$ as

$$z \mapsto z^{-1} \qquad \Longrightarrow \qquad v^{\dot\alpha} \mapsto z^{-1} v^{\dot\alpha}\,. \tag{2.2}$$

So the naive volume form $\mathrm{d}z\,\mathrm{d}^2v$ develops a fourth-order pole at $z = \infty$:

$$z \mapsto z^{-1} \qquad \Longrightarrow \qquad \mathrm{d}z \wedge \mathrm{d}v^{\dot 0} \wedge \mathrm{d}v^{\dot 1} \mapsto -z^{-4}\,\mathrm{d}z \wedge \mathrm{d}v^{\dot 0} \wedge \mathrm{d}v^{\dot 1}\,. \tag{2.3}$$

This tells us that the canonical bundle of $\mathbb{PT}$ is $\mathscr{O}(-4)$, which is *not* trivial.

- Two ways to get around this obstruction have been proposed.

  - Remove the surfaces $z = 0, \infty$ from $\mathbb{PT}$ to obtain a Calabi-Yau with "boundaries". This can be used to engineer partially gauge-fixed formulations of self-dual Yang-Mills (sdYM) that do not have manifest Lorentz symmetry.
  - Consider higher-dimensional Calabi-Yau's fibering over $\mathbb{PT}$ with local descriptions like $\mathscr{O}(-1)\oplus\mathscr{O}(-3) \to \mathbb{PT}$ or $\mathscr{O}(-1)^4 \to \mathbb{PT}$. These can be used to engineer twistor actions for $N = 1$ sdYM or $N = 4$ sdYM respectively on D5 branes wrapping $\mathbb{PT}$. A further orientifold of the former produces non-supersymmetric examples.

  This turns the subject of celestial holography into a systematic model-building exercise!

- We will focus on the first approach which is more pedagogical.

- On $\mathbb{PT} - \{z = 0, \infty\} \simeq \mathbb{C}^\times_z \times \mathbb{C}^2_{v^{\dot\alpha}}$, we choose to work with the volume form

$$\Omega = \frac{\mathrm{d}z \wedge \mathrm{d}v^{\dot 0} \wedge \mathrm{d}v^{\dot 1}}{z^2} \tag{2.4}$$

which symmetrically has a second-order pole at both $z = 0$ and $z = \infty$. We will have to prescribe boundary conditions at the removed divisors $z = 0, \infty$ that keep the target space actions non-singular.

- Precise setup: type I B-model on $\mathbb{PT} - \{z = 0, \infty\}$. This necessarily comes with 8 D5 branes to cancel the charge of the O5 orientifold plane.

- After orientifolding, the physical fields of the remaining spectrum are the Beltrami β in the closed string sector and an SO(8) partial connection a in the open string sector on the 8 D5 branes. There are also ghosts and antifields that we suppress.

- The open string action is

$$S[a] = \int_{\mathbb{PT}} \frac{\mathrm{d}z\, \mathrm{d}^2 v}{z^2}\, \mathrm{Tr}\left(\frac{1}{2}\, a\, \bar\partial a + \frac{1}{3!}\, a^3\right). \tag{2.5}$$

  To ensure that the integral converges, we assume the boundary conditions that a have a first-order zero at $z = 0, \infty$. A similar analysis of the BCOV action shows that the Beltrami β obeys the boundary condition of having a second-order zero at $z = 0, \infty$. Equivalently, the quantity $\beta \lrcorner \Omega$ must be regular at $z = 0, \infty$. These boundary conditions break Lorentz covariance.

- Next, celestial holography motivates studying holography for D1 branes. So, wrap a stack of $2N$ D1 branes along the $\mathbb{C}^\times_z$ direction (they have to be even in number due to the orientifold projection). We will study their worldvolume theory in the next lecture. For now, let us construct a large N duality obtained by sending $N \to \infty$.

- The D1 branes add a source for β to the closed string action,

$$2N \int_{\mathrm{D1}} \partial^{-1}(\beta \lrcorner \Omega)\,. \tag{2.6}$$

  So, as $N \to \infty$, the branes backreact. The backreacted geometry must solve the sourced equation of motion

$$\bar\partial\beta + \frac{1}{2}\,[\beta, \beta] = 2N\, \bar\delta^2(v)\, z^2 \partial_z\,. \tag{2.7}$$

  The factor of z^2 comes from the factor of z^{-2} present in Ω.

- This equation is solved by the Bochner-Martinelli kernel

$$\beta = \frac{2N}{(2\pi i)^2}\, \frac{\mathrm{D}\bar v}{\|v\|^4}\, z^2 \partial_z\,, \tag{2.8}$$

  where $\mathrm{D}\bar v = \varepsilon_{\alpha\beta} \bar v^\alpha \mathrm{d}\bar v^\beta$ and $\|v\|^2 = |v^{\dot 0}|^2 + |v^{\dot 1}|^2$. Its only singularity is at the brane locus $v^{\dot\alpha} = 0$. Note also that no near-horizon limits are necessary in the B-model.

- It turns out that $\mathbb{PT} - \{v^{\dot\alpha} = 0\}$ equipped with the complex structure defined by $\bar\partial + \beta$ is again a (subset of a) twistor space. It is the twistor space of *Burns space*. I will not go into the proof in these lectures, but the interested reader should consult the references. It also has the structure of $\mathrm{AdS}_3 \times S^3$ which leads to a nice holographic interpretation that I will also suppress.

➤ The brane backreaction on twistor space deforms flat space $\mathbb{C}^2$ into Burns space. This is a scalar-flat Kähler 4-manifold which is topologically $\widetilde{\mathbb{C}}^2$: the blow-up of $\mathbb{C}^2$ at the origin. This says that the origin is replaced by a $\mathbb{P}^1$ describing the "direction of approach". But crucially, the region at infinity is still flat $\mathbb{C}^2$.

➤ These properties are best seen from the Burns metric,

$$g = \|\mathrm{d}u\|^2 + \frac{N}{r^4}\,|\mathrm{D}u|^2\,, \tag{2.9}$$

where $r = \|u\|$ and $\mathrm{D}u = \varepsilon_{\dot\alpha\dot\beta}u^{\dot\alpha}\mathrm{d}u^{\dot\beta}$. As $N \to 0$, the backreaction turns off and we recover flat space. For $N > 0$, we find $g_{\mu\nu} \sim \delta_{\mu\nu} + O(r^{-2})$ as $r \to \infty$. So the asymptotic geometry remains flat, exactly as desired. Furthermore, $R = 0$, which confirms that this geometry has no cosmological constant. Unfortunately, $R_{\mu\nu} \neq 0$, so this geometry is not a solution of Einstein gravity.

➤ The backreacted twistor space has the structure of $\widetilde{\mathbb{C}}^2 \times \mathbb{P}^1$, and compactifying the type I B-model on $\mathbb{P}^1$ generates a 4d theory on Burns space.

$$S_{\text{4d}} = S_{\text{Mabuchi}} + S_{\text{WZW}_4} \tag{2.10}$$

– Closed string sector: Mabuchi gravity = Kähler subsector of sd conformal gravity

– D5-D5 open string sector: 4d WZW model based on the group SO(8)

The D1 branes have of course been replaced by the deformed geometry.

➤ The Burns metric is Kähler with Kähler potential

$$K_0 = r^2 + N\log r^2\,. \tag{2.11}$$

The zero mode of the Beltrami gives rise to a spacetime scalar ρ that describes perturbations of the Kähler potential $K_0 \mapsto K = K_0 + \rho$. The gravitational action is

$$S_{\text{Mabuchi}} = \frac{1}{2}\int \mathrm{Ric}(K)\wedge\partial K\wedge\bar\partial K \tag{2.12}$$

whose equation of motion says that the Kähler metric obtained from the deformed potential K continues to be scalar-flat: $R(K) = 0$.

➤ More interesting is the open string sector. The zero mode of the field a gives rise to an $\mathfrak{so}(8)$-valued scalar ϕ with action

$$S_{\text{WZW}_4} = i\int \partial\bar\partial K\wedge\mathrm{Tr}\left(\frac{1}{2}\,\partial\phi\wedge\bar\partial\phi + \frac{1}{3!}\,\partial\phi\wedge[\phi,\bar\partial\phi] + \frac{1}{4!}\,\partial\phi\wedge[\phi,[\phi,\bar\partial\phi]] + O(\phi^5)\right)\,, \tag{2.13}$$

where $\partial = \mathrm{d}u^{\dot\alpha}\partial_{u^{\dot\alpha}}$ and $\bar\partial = \mathrm{d}\bar u^{\alpha}\partial_{\bar u^{\alpha}}$. We can use ϕ to define a gauge field on spacetime,

$$A = -\bar\partial\mathsf{g}\,\mathsf{g}^{-1}\,,\qquad \mathsf{g} = \mathrm{e}^{\phi}\,, \tag{2.14}$$

The field equation of ϕ tells us that A is self-dual on Burns space: $F_A = *F_A$.

3 Lecture 3

➤ In this lecture, we will study the celestial holographic dual and its associated holographic dictionary.

➤ The volume form we used on $\mathbb{PT} - \{z = 0, \infty\}$ was $\Omega = \mathrm{d}z\, \mathrm{d}^2v/z^2 = -\mathrm{d}w\, \mathrm{d}^2v$. So in what follows it will be convenient to change coordinates to $w = z^{-1}$, so that we are essentially working with the B-model and branes in flat space $\mathbb{C}^\times_w \times \mathbb{C}^2_{v^{\dot\alpha}}$.

➤ The $2N$ D1 branes engineer an $\mathrm{Sp}(N)$ gauge theory with gauge-fixed spectrum:

- D1-D1 strings: $\mathcal{A} = c + \mu^{\dot\alpha}\partial_{\dot\alpha} + b\,\partial_{\dot 0} \wedge \partial_{\dot 1}$
  $b, c \in S^2\mathbb{C}^{2N}$ (fermionic), $\mu^{\dot\alpha} \in \wedge^2\mathbb{C}^{2N}$ (bosonic)
- D1-D5 strings: $I \in \mathbb{C}^{2N} \otimes \mathbb{C}^8$ (bosonic)

where $\mathbb{C}^{2N}$ and $\mathbb{C}^8$ denote the fundamental reps of $\mathrm{Sp}(N)$ and $\mathrm{SO}(8)$ respectively. Our gauge choice has again been to drop the D1-D1 fields containing $\mathrm{d}\bar z$.

➤ Let $a, b = 1, \ldots, 2N$ and $i, j = 1, \cdots, 8$ be the Chan-Paton indices of the $\mathbb{C}^{2N}$ and $\mathbb{C}^8$ reps respectively. The index structure on the D1-D1 strings is $c_{ab} = c_{ba}$, $b_{ab} = b_{ba}$ and $\mu^{\dot\alpha}_{ab} = -\mu^{\dot\alpha}_{ba}$, and that on the D1-D5 strings is I_{ai}. The $\mathrm{Sp}(N)$ indices can be raised or lowered using the $\mathrm{Sp}(N)$ invariant symplectic form ε^{ab}, the $\mathrm{SO}(8)$ indices using the Kronecker delta δ^{ij}, and the spinor index $\dot\alpha$ using the 2d Levi-Civita symbol $\varepsilon^{\dot\alpha\dot\beta}$.

➤ In what follows, whenever we invoke $c, \mu^{\dot\alpha}, b$, we will mean the $\mathrm{Sp}(N)$ endomorphisms ${c_a}^b$, ${b_a}^b$ and ${(\mu^{\dot\alpha})_a}^b$ which can be multiplied like ordinary matrices. For example, products like $\mu^{\dot 0}\mu^{\dot 1}$ will mean ${(\mu^{\dot 0}\mu^{\dot 1})_a}^b = {(\mu^{\dot 0})_a}^c{(\mu^{\dot 1})_c}^b$ in components, and $\mu^{\dot\alpha}I$ will mean $(\mu^{\dot\alpha}I)_{ai} = {(\mu^{\dot\alpha})_a}^b I_{bi}$.

➤ Using the general principles discussed in the first lecture, their action is found to be

$$S_{\text{2d}} = \int_{\mathbb{P}^1} \mathrm{d}w \wedge \left(\frac{1}{2}\,\mathrm{Tr}\,\mu_{\dot\alpha}\bar\partial\mu^{\dot\alpha} + \mathrm{Tr}\,b\bar\partial c + \frac{1}{2}\,I^{ai}\bar\partial I_{ai}\right), \tag{3.1}$$

where Tr denotes a trace on staggered $\mathrm{Sp}(N)$ indices, eg., $\mathrm{Tr}\,b\bar\partial c = {b_a}^b\bar\partial {c_b}^a$. The various fields obey free-field OPEs like

$$\mu^{\dot\alpha}_{ab}(w)\,\mu^{\dot\beta}_{cd}(0) \sim \frac{\varepsilon^{\dot\alpha\dot\beta}\varepsilon_{a[c|}\varepsilon_{b|d]}}{w}, \qquad I_{ai}(w)\,I_{bj}(0) \sim \frac{\varepsilon_{ab}\delta_{ij}}{w}. \tag{3.2}$$

With our gauge choice, this chiral CFT comes equipped with a BRST charge,

$$\begin{aligned} &Q_B c = c^2\,, \qquad Q_B\mu^{\dot\alpha} = [c, \mu^{\dot\alpha}]\,, \\ &Q_B b = [c, b] + [\mu^{\dot 0}, \mu^{\dot 1}] + \frac{1}{2}\,I^i I_i\,, \qquad Q_B I = cI\,, \end{aligned} \tag{3.3}$$

which implements the $\mathrm{Sp}(N)$ gauging. The $\mathrm{SO}(8)$ symmetry remains a flavor symmetry.

➤ The large N BRST cohomology is spanned by three towers of "single trace" states:

– Open string states:

$$J_{ij}[k,l] = \varepsilon(I_i, \mu_{\dot 0}^{(k}\mu_{\dot 1}^{l)} I_j) \equiv \varepsilon^{ab} I_{ai} (\mu_{\dot 0}^{(k}\mu_{\dot 1}^{l)} I)_{bj} \tag{3.4}$$

– Closed string states:

$$\begin{aligned} E[k,l] &= \mathrm{Tr}\, \mu_{\dot 0}^{(k}\mu_{\dot 1}^{l)}\,, \\ F[k,l] &= \mathrm{Tr}\, \mu_{\dot\alpha}\partial\mu^{\dot\alpha} \mu_{\dot 0}^{(k}\mu_{\dot 1}^{l)} + \text{ghost corrections}\,, \end{aligned} \tag{3.5}$$

where $\mu_{\dot 0}^{(k}\mu_{\dot 1}^{l)}$ denotes the symmetrized product of k $\mu_{\dot 0}$ matrices and l $\mu_{\dot 1}$ matrices.

➤ As a simple example, consider the current $J[0,0]$:

$$\begin{aligned} Q_B J_{ij}[0,0] = Q_B(I_{ai} I^a_j) &= c_a{}^b I_{bi} I^a_j + I_{ai} c^{ab} I_{bj} \\ &= c^{ab}(I_{ai} I_{bj} - I_{bi} I_{aj}) = 0\,, \end{aligned} \tag{3.6}$$

having used $c_a{}^b I^a_j = -c^{ab} I_{aj}$ and $c^{ab} = c^{ba}$.

➤ For each such operator, we can locate a gluon or graviton state in the bulk. The $J[k,l]$ are S-algebra currents:

$$J_{ij}[k,l](w)\, J_{kl}[m,n](0) \sim \frac{f_{ij,kl}{}^{pq} J_{pq}[k+m, l+n]}{w}\,. \tag{3.7}$$

So we expect them to be dual to soft modes of positive helicity gluons described by a scalar profile ϕ on Burns space solving its linearized field equation $\triangle\phi = 0$.

➤ The $E[k,l]$ get associated to positive helicity Einstein gravitons, and the $F[k,l]$ with positive helicity conformal gravitons, and both types of gravitons are described by scalar linear fields ρ.

➤ Weirdly, the stress tensor is a conformal graviton!

$$F[0,0] = \mathrm{Tr}\left(\frac{1}{2}\,\mu_{\dot\alpha}\partial\mu^{\dot\alpha} + b\partial c\right) + \frac{1}{2}\, I^{ai}\partial I_{ai}\,. \tag{3.8}$$

Unclear how to relate this to the usual shadow prescription of a negative helicity Einstein graviton. Perhaps the w-symmetry is more fundamental than BMS symmetry...

➤ In what follows, we focus on gluon states. To compute amplitudes, it is convenient to work with hard states. This motivates introducing the exponential-type operator

$$J(w, \widetilde\lambda) = \varepsilon(I, \mathrm{e}^{i[\mu\widetilde\lambda]} I)\,, \qquad [\mu\widetilde\lambda] \equiv \mu_{\dot\alpha}\widetilde\lambda^{\dot\alpha}\,, \tag{3.9}$$

where $\widetilde\lambda_{\dot\alpha} \in \mathbb{C}^2$. Expanding this in $\widetilde\lambda^{\dot\alpha}$ gives rise to the S-algebra currents $J[k,l]$.

➤ Define a complex null momentum $p_{\alpha\dot\alpha} = \lambda_\alpha\widetilde\lambda_{\dot\alpha}$, $\lambda_\alpha = (1,z) = (1,w^{-1})$. Recall the coordinates $u^{\dot\alpha}$ on Burns space, and define $\hat u^{\dot\alpha} = (-\bar u^1, \bar u^0)$. Then this operator is found to be dual to a plane-wave-like state on Burns space:

$$\phi = \mathfrak{t}_{ij}\left(\mathrm{e}^{ip\cdot x} - \frac{Nz[u\widetilde\lambda][\hat u\widetilde\lambda]}{r^2}\, {}_1F_1(2,3|ip\cdot x) + O(N^2)\right) \tag{3.10}$$

where the corrections (which can be found in closed form) ensure that ϕ solves $\triangle\phi = 0$ with respect to the Burns metric. Here $\mathfrak{t}_{ij} = -\mathfrak{t}_{ji}$ is a generator of $\mathfrak{so}(8)$, and the associated positive helicity gluon wavefunction is given by $A = -\bar{\partial}\phi$.

- The holographic conjecture says that correlators of such operators $J(w_i, \widetilde{\lambda}_i)$, $i = 1, \ldots, n$, will compute n-point amplitudes of the 4d WZW model on Burns space. At tree level, these will coincide with all-plus gluon amplitudes on Burns space.

- We can study this conjecture using the 2-point amplitude which was computed by Hawking, Page and Pope in 1980. Our bulk theory doesn't have manifest conformal symmetry so even this match is somewhat non-trivial.

- HPP compute a 2-point scalar amplitude, which can be dressed with color-factors to produce the $++$ gluon tree amplitude

$$A(1,2) = \frac{N\,\mathrm{Tr}(\mathfrak{t}_{ij}\mathfrak{t}_{kl})}{w_{12}^2}\, J_0\left(\sqrt{4N\,\frac{[12]}{w_{12}}}\right) \tag{3.11}$$

where $[12] = [\widetilde{\lambda}_1\widetilde{\lambda}_2]$ and $w_{12} = w_1 - w_2$. Let us show that this can be computed by the JJ 2-point function.

- Let's compute the 2-point correlator

$$\left\langle J_{ij}(w_1,\widetilde{\lambda}_1)\, J_{kl}(w_2,\widetilde{\lambda}_2)\right\rangle = \sum_{m,n\geq 0} \frac{i^{m+n}}{m!\,n!} \left\langle \varepsilon(I_i, [\mu 1]^m I_j)(w_1)\, \varepsilon(I_k, [\mu 2]^n I_l)(w_2)\right\rangle . \tag{3.12}$$

Since all operators need to be contracted away to produce a non-zero 2-point function, only terms with $m = n$ can contribute. So we get

$$\left\langle J_{ij}(w_1,\widetilde{\lambda}_1)\, J_{kl}(w_2,\widetilde{\lambda}_2)\right\rangle = \sum_{n\geq 0} \frac{(-1)^n}{(n!)^2} \left\langle \varepsilon(I_i, [\mu 1]^n I_j)(w_1)\, \varepsilon(I_k, [\mu 2]^n I_l)(w_2)\right\rangle . \tag{3.13}$$

Note that

$$[\mu_a{}^b\, 1](w_1)\, [\mu_c{}^d\, 2](w_2) \sim \frac{\varepsilon_{ac}\varepsilon^{bd} - \delta_a^d\delta_c^b}{2}\, \frac{[12]}{w_{12}} . \tag{3.14}$$

Keeping only planar double-line diagrams gives the desired tree amplitude:

$$\begin{aligned} \left\langle J_{ij}(w_1,\widetilde{\lambda}_1)\, J_{kl}(w_2,\widetilde{\lambda}_2)\right\rangle &= \sum_{n\geq 0} \frac{(-1)^n}{(n!)^2}\, \frac{\mathrm{Tr}(\mathfrak{t}_{ij}\mathfrak{t}_{kl})}{w_{12}^2}\, \frac{N^{n+1}[12]^n}{w_{12}^n} \\ &= \frac{N\,\mathrm{Tr}(\mathfrak{t}_{ij}\mathfrak{t}_{kl})}{w_{12}^2}\, J_0\left(\sqrt{4N\,\frac{[12]}{w_{12}}}\right) . \end{aligned} \tag{3.15}$$

Similarly, we can obtain higher point amplitudes.