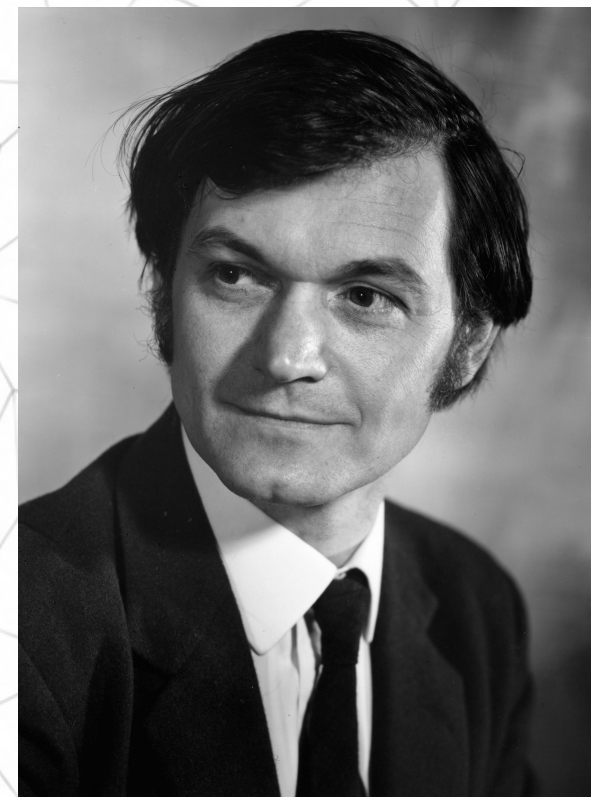


The Giants of Twistor Theory

Atul Sharma

Black Hole Initiative, Harvard

Happy 95th!!!



Twistor Algebra

R. PENROSE

Department of Mathematics, Birkbeck College, London, England

(Received 16 May 1966)

A new type of algebra for Minkowski space-time is described, in terms of which it is possible to express any conformally covariant or Poincaré covariant operation. The elements of the algebra (twistors) are combined according to tensor-type rules, but they differ from tensors or spinors in that

Twistor theory highly successful for describing classical gravity and gauge theory.

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Twistor theory highly successful for describing classical gravity and gauge theory.

Can we use it to model aspects of **quantum gravity**?

Questions for Today

- How does quantum gravity “**quantize**” features of semiclassical gravity?
- What is the **nonperturbative** gravitational path integral at finite $\hbar$?

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- How does quantum gravity “**quantize**” features of semiclassical gravity?
- What is the **nonperturbative** gravitational path integral at finite $\hbar$?

We will study these questions through solvable models of
holography in twistor space.



$$\text{Entropy} = \frac{\text{Area}}{4 \ell_{\text{P}}^{d-1}}$$

AdS/CFT duality



Gravity in AdS_d



Rank N gauge theory
on ∂AdS_{d-1}

AdS/CFT suggests

$$\ell_{\text{P}}^{d-1} \sim \frac{\ell_{\text{AdS}}^{d-1}}{N^2}$$

$$\text{Entropy} \sim \text{O}(N^2)$$

N = degrees of freedom of boundary theory

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N = degrees of freedom of boundary theory

In quantum gravity, we cannot resolve spacetime on scales smaller than $1/N$

Canonical example of (Euclidean) AdS/CFT

Type IIB string
theory on
 $H_5 \times S^5$



$U(N)$ $\mathcal{N} = 4$
super Yang-Mills
on $\partial H_5 = S^4$

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Fields of boundary theory: for $I, J = 1, 2, 3, 4$

$$A_a, \Psi_\alpha^I, \widetilde{\Psi}_{I\dot{\alpha}}, \Phi_{[IJ]}$$

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$$A_a, \Psi_\alpha^I, \widetilde{\Psi}_{I\dot{\alpha}}, \Phi_{[IJ]}$$

valued in $\mathfrak{u}(N)$

**States of the
bulk theory**



**Operators of the
boundary theory**

States of the
bulk theory



Operators of the
boundary theory

Semiclassical gravity : $N \rightarrow \infty$

Φ an $N \times N$ matrix

$\text{Tr } \Phi, \text{Tr } \Phi^2, \text{Tr } \Phi^3, \dots$

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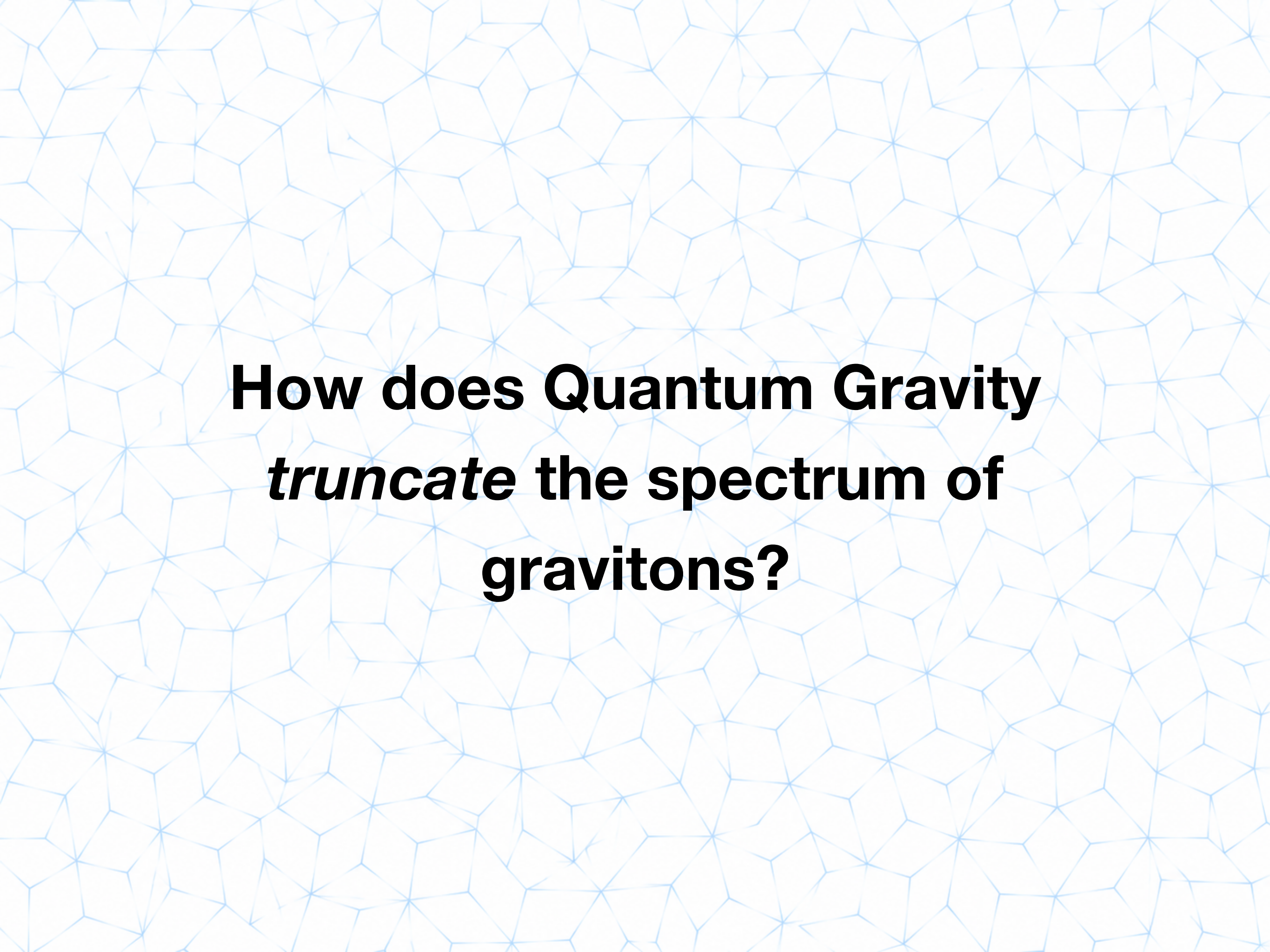
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Quantum gravity : **Cayley-Hamilton** truncates this to

$\text{Tr } \Phi, \text{Tr } \Phi^2, \text{Tr } \Phi^3, \dots, \text{Tr } \Phi^N$



**How does Quantum Gravity
truncate the spectrum of
gravitons?**

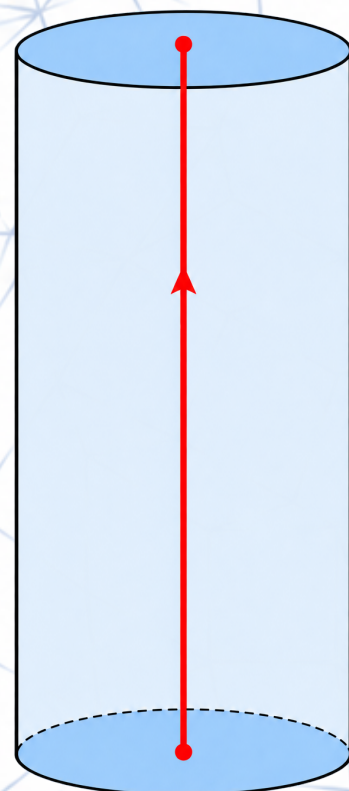
Invasion of the giant gravitons from anti-de Sitter space

John McGreevy, Leonard Susskind and Nicolaos Toumbas

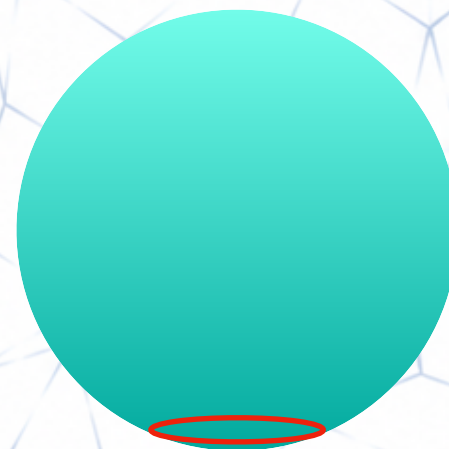
Department of Physics, Stanford University

Stanford CA 94305-4060, USA

E-mail: mcgreevy@stanford.edu, susskind@sewerrat.stanford.edu,
ntoumbas@leland.stanford.edu



AdS_5



S^5

$\text{Tr } \Phi$

Invasion of the giant gravitons from anti-de Sitter space

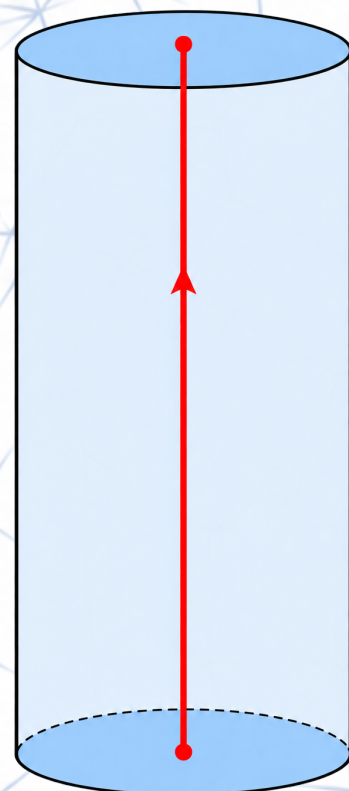
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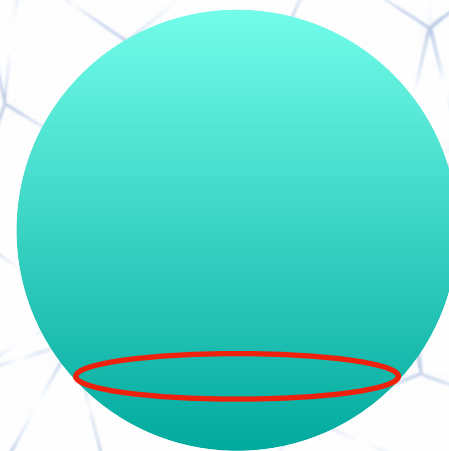
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ntoumbas@leland.stanford.edu

JHEP06



AdS_5



S^5

$$\text{Tr } \Phi^2 - (\text{Tr } \Phi)^2$$

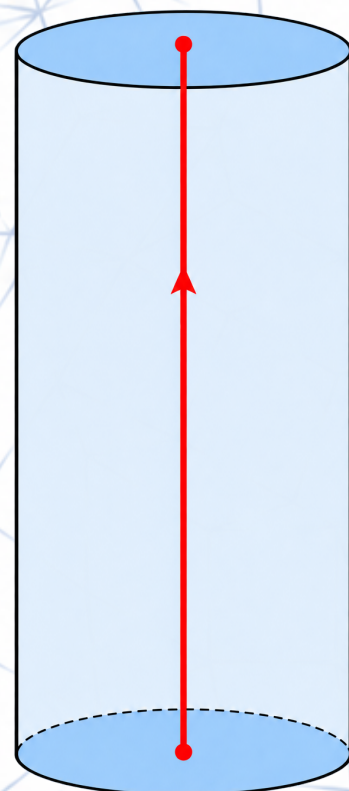
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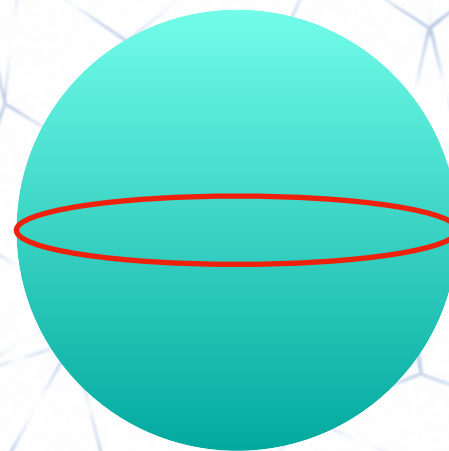
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AdS_5



S^5

$$\det \Phi = \text{Tr} \Phi^N + \dots$$

Invasion of the giant gravitons from anti-de Sitter space

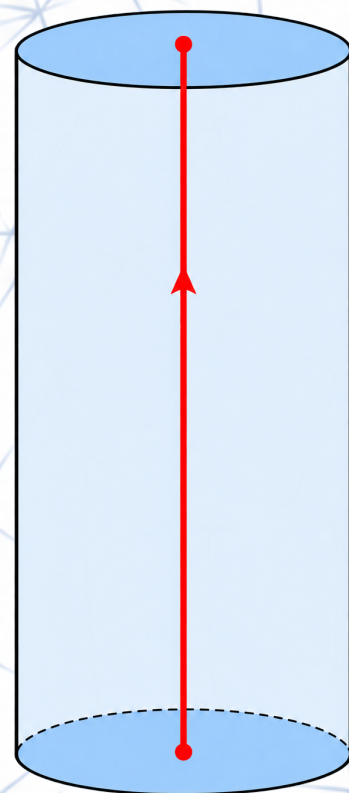
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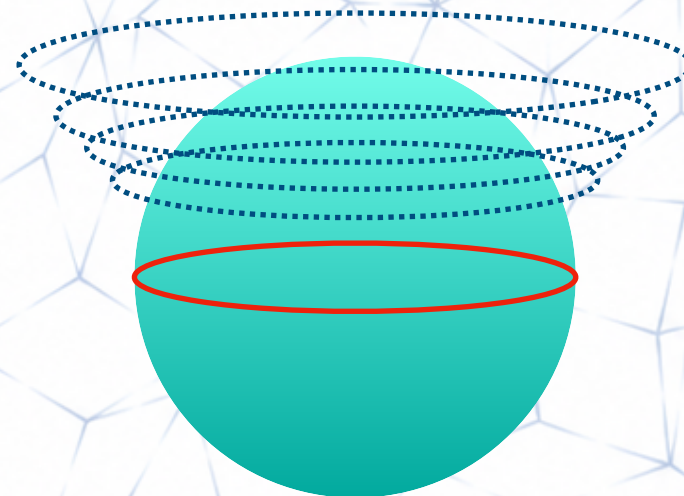
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JHEP06



AdS_5



S^5

Overmaximal
giant gravitons

Maximal giant
graviton

The Inclusion/Exclusion principle

Bulk

Gravitons
at finite N

=

Gravitons at
infinite N

-

Overmaximal
gravitons



Boundary

Traces at
finite N

=

Traces at
infinite N

-

Trace
relations

The Inclusion/Exclusion principle

Bulk

Gravitons at finite N = Gravitons at infinite N - Overmaximal gravitons



Boundary

Traces at finite N = Traces at infinite N - Trace relations

Quantum = Perturbative + Nonperturbative

**Counting
multi-traces**

$$(\mathrm{Tr} \Phi)^{k_1} (\mathrm{Tr} \Phi^2)^{k_2} (\mathrm{Tr} \Phi^3)^{k_3} \dots (\mathrm{Tr} \Phi^N)^{k_N}$$

$$I_N(q) = \prod_{n=1}^N \frac{1}{1 - q^n}$$

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Giant graviton expansion

[Imamura '21] [Gaiotto, Lee '21]

$$I_N(q) = I_\infty(q) \sum_{m=0}^{\infty} q^{mN} G_m(q)$$

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Subtracts
overmaximal giants

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$$G_m(q) = \prod_{n=1}^m \frac{1}{1 - q^{-n}} = \text{counts } m \text{ overmaximal giants}$$

Recent advances: **holography in twistor space**

[Costello, Paquette, **AS** '22]

[Zeng '23]

[Bittleston, Costello, Zeng '24]

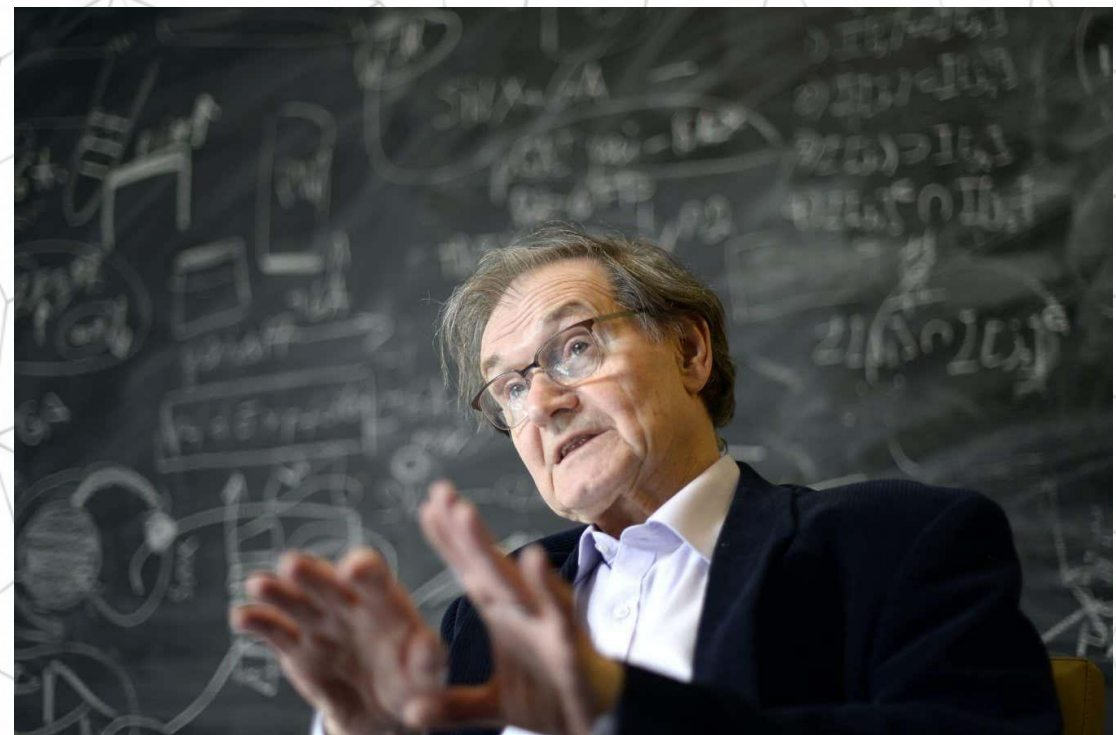
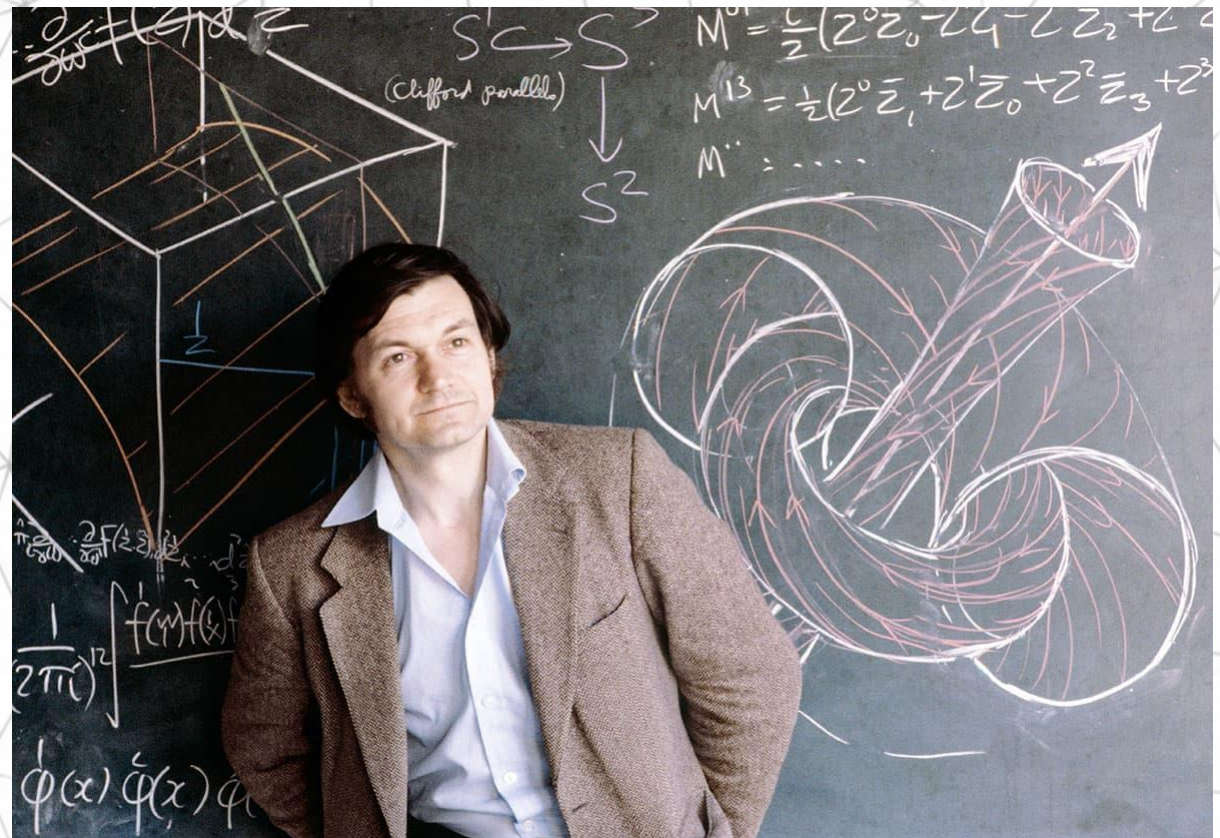
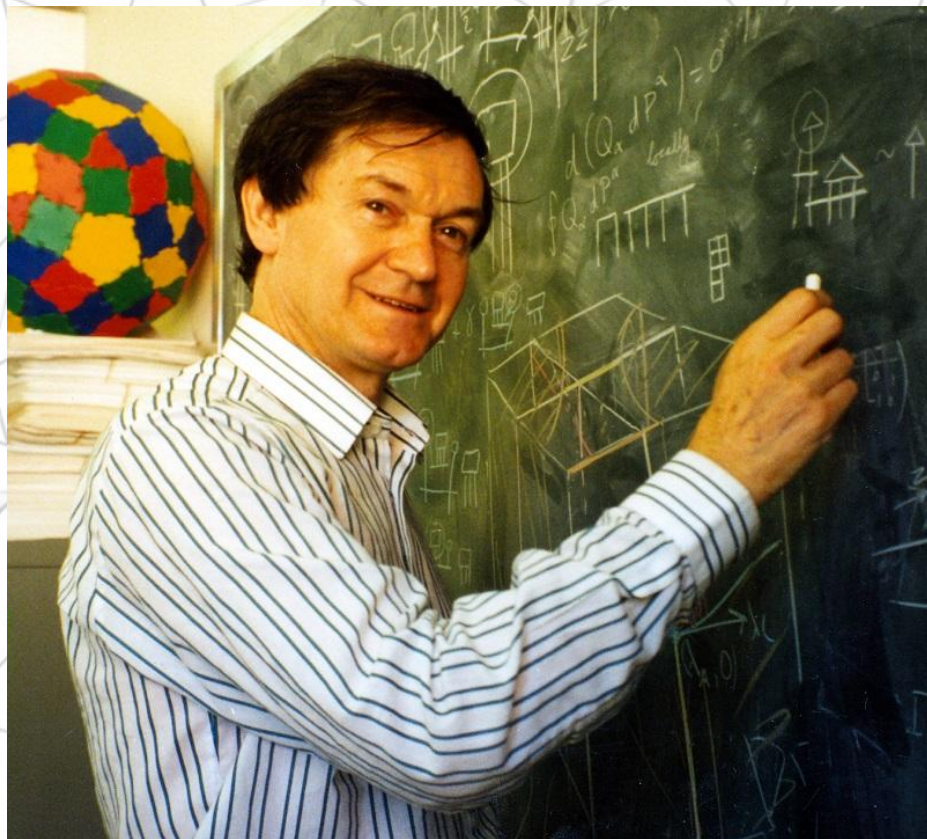
[**AS**, Skinner '25]

[Jarov '25]

[Balisa, Casali '26]

Can we understand finite- N effects via **twistorial holography**?

Biggest GIANT in twistor theory



Why use twistors at all?

AdS/CFT is a **strong/weak** duality.

Boundary CFT

Bulk gravity

$1/N$ expansion



Loop expansion

$g_{\text{YM}}^2 N$ expansion



Expansion in
string tension

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Expansion in
string tension

Non-stringy gravity corresponds to $g_{\text{YM}}^2 N \rightarrow \infty$.

Usually impossible to calculate at strong 't Hooft coupling!

Why use twistors at all?

Usual cop-out: compute quantities that do not depend on coupling

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Examples:

- $1/2$, $1/8$ or $1/16$ BPS index
- 2, 3-point functions of BPS operators

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Examples:

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- 2, 3-point functions of BPS operators

But tests of holography beyond such **susy-protected** quantities remain obstructed!

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Standard AdS/CFT works with full super YM.

Can we instead find a gravity dual of **self-dual** super YM?

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Penrose-Ward
Correspondence

+

Witten's
Twistor String

=

**Chiral
Holography**

Yes!

[AS, Skinner '25]

General Relativity and Gravitation, Vol. 7, No. 1 (1976), pp. 31–52

Nonlinear Gravitons and Curved Twistor Theory

ROGER PENROSE

Mathematical Institute, Oxford, England

Volume 61A, number 2

PHYSICS LETTERS

18 April 1977

ON SELF-DUAL GAUGE FIELDS

R.S. WARD

Mathematical Institute, University of Oxford, UK

Received 26 February 1977

Twistor String Theory

B-model topological string on the Calabi-Yau 7-fold

$$X = \text{Tot}(\mathcal{O}(-1)^{\oplus 4} \rightarrow \mathbb{P}\mathbb{T})$$

[Witten '03]

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Open twistor strings on N D-branes wrapping $\mathbb{P}\mathbb{T}$ have spectrum

$$\bigoplus_{p=0}^4 H^1(\mathbb{P}\mathbb{T}, \wedge^p \mathcal{N}_{\mathbb{P}\mathbb{T}/X}) \otimes \mathfrak{gl}_N(\mathbb{C})$$

States of $\mathcal{N} = 4$
vector multiplet

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Penrose-Ward transform gives $U(N)$ $\mathcal{N} = 4$ **SD**YM on $\mathbb{R}^4$

Relation to supertwistor space $\mathbb{PT}^{3|4}$

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Coordinates :

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Open twistor
string field : $\mathcal{A} = a + \widetilde{\psi}_I \chi^I + \frac{1}{2} \varphi_{IJ} \chi^I \chi^J + \psi^I \chi_I^3 + b \chi^4$

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$\swarrow$ $A_{\alpha\dot{\alpha}}$ $\swarrow$ $\tilde{\Psi}_{I\dot{\alpha}}$ $\downarrow$ Φ_{IJ} $\swarrow$ Ψ^I_α $\swarrow$ $B_{\alpha\beta}$

Chiral Holography

In the large N limit, replace D5s by their backreaction.

$$\begin{array}{l} \text{U}(N) \mathcal{N} = 4 \\ \text{SDYM on } \mathbb{R}^4 \end{array} = \begin{array}{l} \text{Closed twistor string theory} \\ + N \text{ units of backreaction} \end{array}$$

[AS, Skinner '25]

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[AS, Skinner '25]

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Can finally perform non-BPS tests of holography!

Target: non-BPS SDYM partition function at finite N

$$Z_N = \text{Tr}_{\mathcal{H}_N} (-1)^F x_1^{-R_1} x_2^{-R_2} x_3^{-R_3} x_4^{-R_4} y_1^{-J_1} y_2^{-J_2} z_1^{-L_1} z_2^{-L_2}$$

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Fugacity space: $[x_1 : x_2 : x_3 : x_4 : y_1 : y_2 : z_1 : z_2] \in \mathbb{P}^7$

$$x_1 x_2 x_3 x_4 = y_1 y_2 z_1 z_2$$

Reduces $\text{GL}(4 | 4, \mathbb{C})$ to $\text{PSL}(4 | 4, \mathbb{C})$

Unrefined specialization

$$x_I = 1 \qquad y_1 = y_2 = t \qquad z_1 = z_2 = t^{-1}$$

$$Z_N = \text{Tr}_{\mathcal{H}_N} (-1)^F t^{2D}$$

$$D = \frac{1}{2} \left(\lambda_\alpha \frac{\partial}{\partial \lambda_\alpha} - \mu^{\dot{\alpha}} \frac{\partial}{\partial \mu^{\dot{\alpha}}} \right)$$

SDYM path integral on $S^1 \times S^3$ is 1-loop exact.

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So we simply count states of linearized SDYM & project onto gauge invariants.

More precisely, compute the cohomology of BV differential

$$Q\mathcal{A} = \bar{\partial}\mathcal{A} + \mathcal{A}^2$$

via a spectral sequence and Lie algebra cohomology.

Linear theory states can be counted via Penrose transform.

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 10, NUMBER 1 JANUARY 1969

Solutions of the Zero-Rest-Mass Equations

ROGER PENROSE

Birkbeck College, London, England

(Received 23 September 1966)

By means of contour integrals involving arbitrary analytic functions, general solutions of the zero-rest-mass field equations in flat space-time can be generated for each spin. If the contour surrounds only a simple (respectively, low-order) pole of the function, the resulting field is null (respectively, algebraically special).

Cohomology and Massless Fields[★]

Michael G. Eastwood¹, Roger Penrose¹, and R. O. Wells², Jr.

¹ Mathematical Institute, Oxford University, Oxford OX1 3LB, England, and

² Department of Mathematics, Rice University, Houston, TX 77001, USA

Expansion in Čech representatives

$$\begin{aligned}
 \mathcal{A} = & \sum_{p_1=0}^1 \sum_{p_2=0}^1 \sum_{p_3=0}^1 \sum_{p_4=0}^1 \sum_{q_1=0}^{\infty} \sum_{q_2=0}^{\infty} \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} \mathbf{a}^{\dagger}[\mathbf{p}, \mathbf{q}, \mathbf{r}] \\
 & \times \lambda_1^{-1-r_1} \lambda_2^{-1-r_2} (\mu^{\dot{1}})^{q_1} (\mu^{\dot{2}})^{q_2} (\chi^1)^{p_1} (\chi^2)^{p_2} (\chi^3)^{p_3} (\chi^4)^{p_4} \\
 & \times \delta_{p_1+p_2+p_3+p_4+q_1+q_2, r_1+r_2+2}
 \end{aligned}$$

Expansion in Čech representatives

$\mathfrak{gl}_N(\mathbb{C})$ -valued
Creation Ops

$$\begin{aligned} \mathcal{A} = & \sum_{p_1=0}^1 \sum_{p_2=0}^1 \sum_{p_3=0}^1 \sum_{p_4=0}^1 \sum_{q_1=0}^{\infty} \sum_{q_2=0}^{\infty} \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} \boxed{\mathbf{a}^\dagger[\mathbf{p}, \mathbf{q}, \mathbf{r}]} \\ & \times \lambda_1^{-1-r_1} \lambda_2^{-1-r_2} (\mu^{\dot{1}})^{q_1} (\mu^{\dot{2}})^{q_2} (\chi^1)^{p_1} (\chi^2)^{p_2} (\chi^3)^{p_3} (\chi^4)^{p_4} \\ & \times \delta_{p_1+p_2+p_3+p_4+q_1+q_2, r_1+r_2+2} \end{aligned}$$

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Before imposing gauge invariance, a general state is a product of creation operators acting on the vacuum.

$$\mathbf{a}^\dagger[\mathbf{p}, \mathbf{q}, \mathbf{r}] \cdots \mathbf{a}^\dagger[\mathbf{p}', \mathbf{q}', \mathbf{r}'] | 0 \rangle$$

Fugacity
assignment:

$$[\mathbf{a}^\dagger[\mathbf{p}, \mathbf{q}, \mathbf{r}]] = x_1^{p_1} x_2^{p_2} x_3^{p_3} x_4^{p_4} y_1^{q_1} y_2^{q_2} z_1^{-1-r_1} z_2^{-1-r_2}$$

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Graded Hilbert series of cohomology of $\mathbb{P}\mathbb{T}^{3|4}$

$$f(x, y, z) = \sum_{p_1=0}^1 \cdots \sum_{r_2=0}^{\infty} (-1)^{p_1+p_2+p_3+p_4} [\mathbf{a}^\dagger[\mathbf{p}, \mathbf{q}, \mathbf{r}]] \delta_{p_1+p_2+p_3+p_4+q_1+q_2, r_1+r_2+2}$$

Fugacity
assignment:

$$[\mathbf{a}^\dagger[\mathbf{p}, \mathbf{q}, \mathbf{r}]] = x_1^{p_1} x_2^{p_2} x_3^{p_3} x_4^{p_4} y_1^{q_1} y_2^{q_2} z_1^{-1-r_1} z_2^{-1-r_2}$$

Graded Hilbert series of cohomology of $\mathbb{P}\mathbb{T}^{3|4}$

$$f(x, y, z) = \sum_{p_1=0}^1 \cdots \sum_{r_2=0}^{\infty} (-1)^{p_1+p_2+p_3+p_4} [\mathbf{a}^\dagger[\mathbf{p}, \mathbf{q}, \mathbf{r}]] \delta_{p_1+p_2+p_3+p_4+q_1+q_2, r_1+r_2+2}$$

$$= 1 - \frac{z_1 P(z_1) - z_2 P(z_2)}{z_1 - z_2}$$

$$P(z) = \frac{(1 - z^{-1}x_1)(1 - z^{-1}x_2)(1 - z^{-1}x_3)(1 - z^{-1}x_4)}{(1 - z^{-1}y_1)(1 - z^{-1}y_2)}$$

Graded partition function of $\mathcal{N} = 4$ SDYM

$$Z_N = \int_{U(N)} dU \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} f(x^n, y^n, z^n) \text{Tr}(U^n) \text{Tr}(U^{-n}) \right)$$

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Unrefined version

$$Z_N = \int_{U(N)} dU \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} f(t^n) \text{Tr}(U^n) \text{Tr}(U^{-n}) \right)$$

$$f(t) = \frac{6t^2 + 2t^3}{(1+t)^3}$$

$$Z_{\infty} = 1 + 6t^2 - 16t^3 + 72t^4 - 240t^5 + 928t^6 - 3456t^7 + \dots$$

- ♦ $O(t^2) : \text{Tr}(\Phi_{IJ})$
- ♦ $O(t^3) : \text{Tr}(\Psi_{\alpha}^I), \text{Tr}(\widetilde{\Psi}_{I\dot{\alpha}})$
- ♦ $O(t^4) : \text{Tr}(D_{\alpha\dot{\alpha}}\Phi_{IJ}), \text{Tr}(B_{\alpha\beta}), \text{Tr}(F_{\dot{\alpha}\dot{\beta}}), \text{Tr}(\Phi_{IJ}\Phi_{KL}), \text{Tr}(\Phi_{IJ})\text{Tr}(\Phi_{KL})$
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- ⋮

Main result at finite N

$$Z_N = Z_{\text{closed}} \sum_{m_0, m_1=0}^{\infty} G_N^{(m_0, m_1)}$$

[Budzik, AS wip]

Proof technique.

Borodin-Okounkov (1999) + Murthy (2022)

Express Z_N as a Hubbard-Stratonovich transform of the τ -function of an auxiliary free fermion 2d CFT. Then use Borodin-Okounkov's expansion for the τ -function.

Ingredients

$$Z_N = Z_{\text{closed}} \sum_{m_0, m_1=0}^{\infty} G_N^{(m_0, m_1)} \quad [\text{Budzik, AS wip}]$$

$$Z_{\text{closed}} = \text{PE}[S(z_1)] \text{PE}[S(z_2)]$$

$$S(z) = \frac{x_1}{z - x_1} + \frac{x_2}{z - x_2} + \frac{x_3}{z - x_3} + \frac{x_4}{z - x_4} - \frac{y_1}{z - y_1} - \frac{y_2}{z - y_2}$$

$$\text{PE}[f(x, y, z)] = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} f(x^n, y^n, z^n)\right)$$

Ingredients

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$$G_N^{(m_0, m_1)} = \int_{U(m_0|m_0)} d\mathcal{W}_0 (\text{Ber } \mathcal{W}_0)^N \int_{U(m_1|m_1)} d\mathcal{W}_1 \text{PE} \left[\sum_{a,b=0}^1 \mathbf{f}'_{ab} \text{Str}(\mathcal{W}_a^{-1}) \text{Str}(\mathcal{W}_b) \right]$$

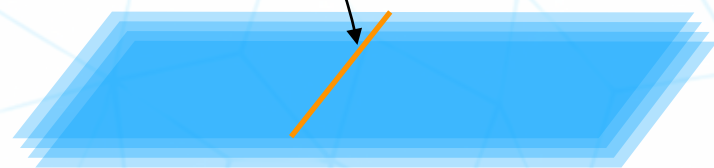
$$\mathbf{f}' = \begin{pmatrix} 1 - \frac{z_1 P(z_2)^{-1} - z_2 P(z_1)^{-1}}{z_1 - z_2} & \frac{P(z_2)^{-1} - P(z_1)^{-1}}{z_1 - z_2} \\ \frac{z_1^2 P(z_2)^{-1} - z_2^2 P(z_1)^{-1}}{z_1 - z_2} & 1 - \frac{z_1 P(z_2)^{-1} - z_2 P(z_1)^{-1}}{z_1 - z_2} \end{pmatrix}$$

Gravity interpretation

Z_{closed} : Counts **closed twistor strings**

$G_N^{(m_0, m_1)}$: Counts Witten's **D-instantons** with Chan-Paton super vector bundle $\mathcal{O}^{m_0|m_0} \oplus \mathcal{O}(1)^{m_1|m_1}$

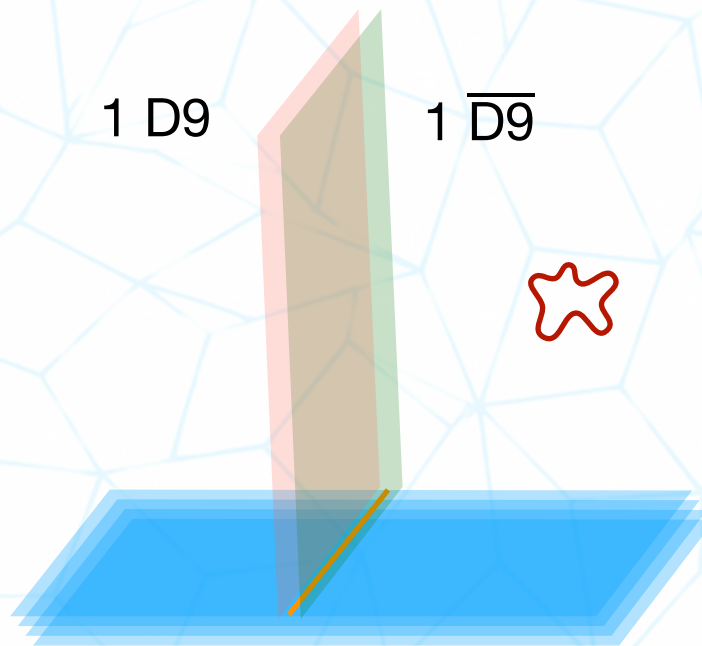
Twistor line
of $0 \in S^4$



$$m_0 = m_1 = 0$$

1 D9

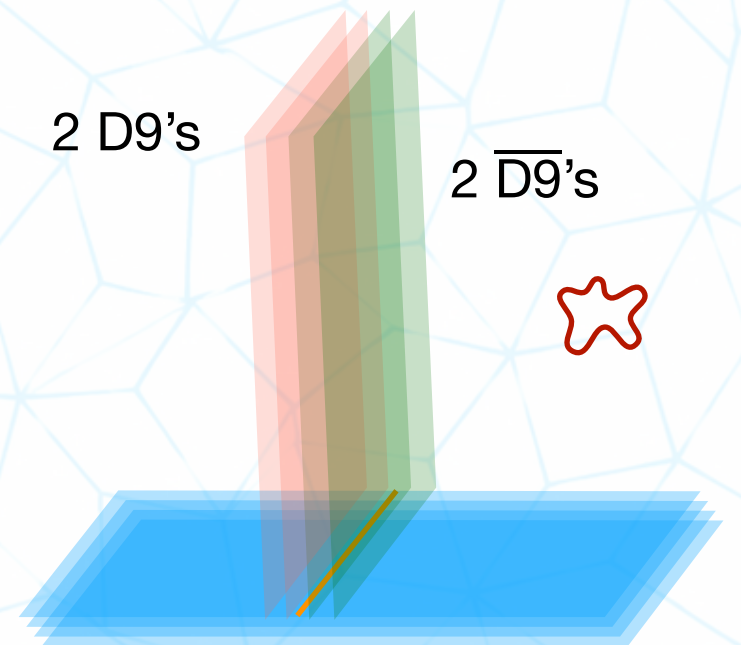
1 $\overline{\text{D9}}$



$$m_0 + m_1 = 1$$

2 D9's

2 $\overline{\text{D9}}$'s



$$m_0 + m_1 = 2$$

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BCOV theory on $X = \text{Tot}(\mathcal{O}(-1)^{\oplus 4} \rightarrow \mathbb{P}^1) + \text{brane backreaction}$

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Deforms states but does not affect their counting.

Closed string states counted by local cohomology of X

$$\bigoplus_{k,l} H_Y^{0,l}(X, \Omega_{\text{exact}}^{k,0})$$

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Counting such states produces $Z_{\text{closed}} = \text{PE}[S(z_1)] \text{PE}[S(z_2)]$

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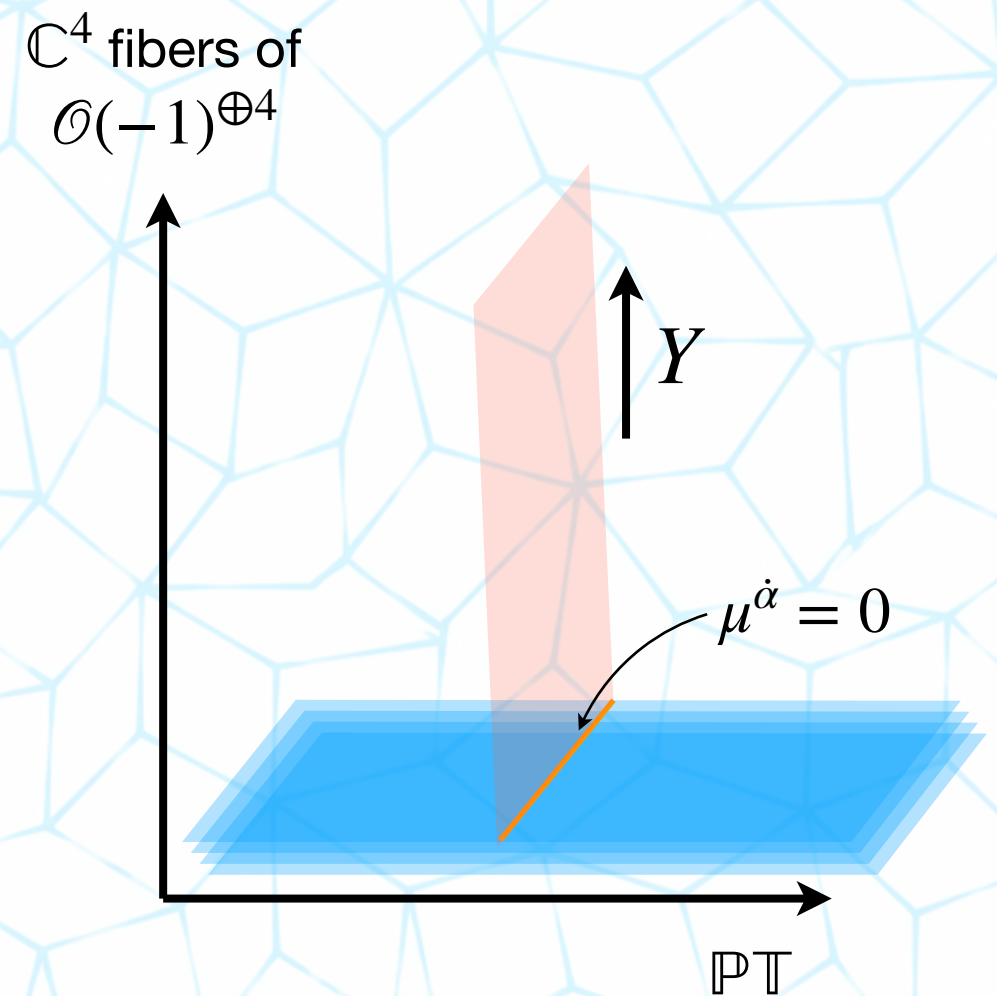
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Fact: This is generated by $\mathcal{O}_X, \mathcal{O}_X(1)$.

[Beilinson '78]

D-instanton corrections

General expectation: suffices to consider branes contained within $Y = \{\mu^{\dot{\alpha}} = 0\} \subset X$. [Budzik '25]

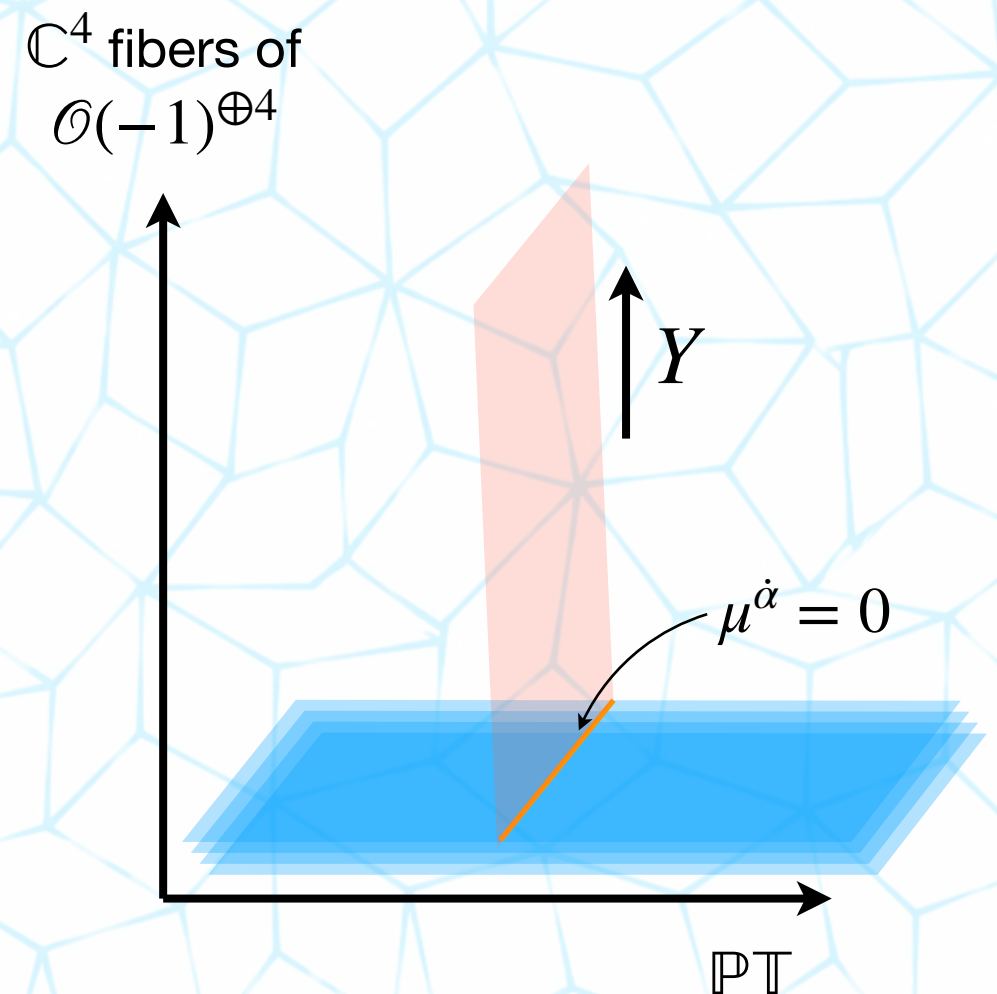


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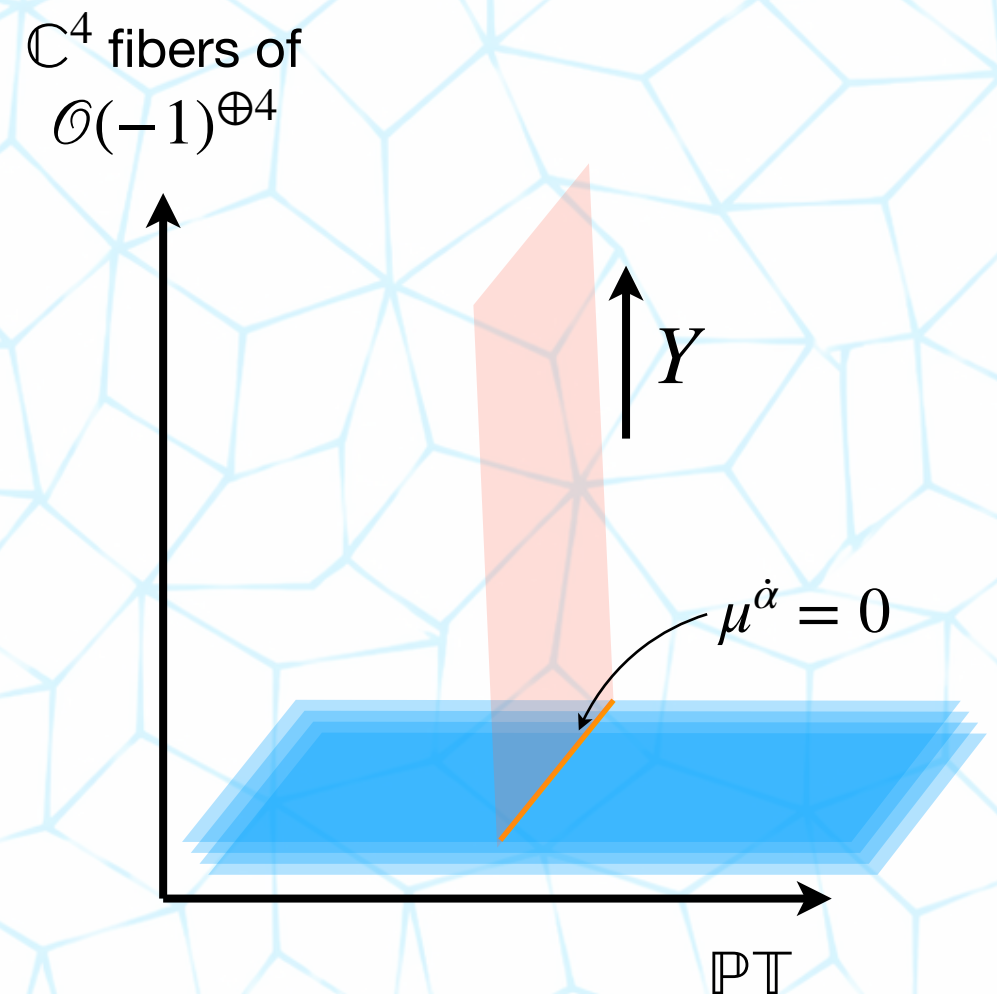
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Call these D-instantons, though Witten's original D-instantons were $\mathcal{O}_Y(1)$



Tachyon condensation: General branes constructed as cohomology of complexes built out of direct sums of $\mathcal{O}_Y$, $\mathcal{O}_Y(1)$.

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m_0 D9 branes and m_0 anti-D9 branes with bundle $\mathcal{O}_Y$

m_1 D9 branes and m_1 anti-D9 branes with bundle $\mathcal{O}_Y(1)$

Eg. “Giant graviton” dual to $\det \Phi_{12}$

$\mathcal{O}_G$: structure sheaf of $G = \{W_1 = W_2 = 0\} \subset Y$ [AS, Skinner '25]

$$\begin{array}{ccccccc}
 & & \mathcal{O}_Y(1) & \xrightarrow{-\lambda_2 W_2} & \mathcal{O}_Y(1) & & \\
 & \nearrow \lambda_1 & & \searrow \lambda_2 W_1 & \nearrow & & \\
 0 & \longrightarrow & \mathcal{O}_Y & \oplus & \mathcal{O}_Y & \longrightarrow & 0 \\
 & \searrow \lambda_2 & & \nearrow \lambda_1 W_2 & \searrow & & \\
 & & \mathcal{O}_Y(1) & \xrightarrow{-\lambda_1 W_1} & \mathcal{O}_Y(1) & & \\
 & & & & & \nearrow W_2 & \\
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 \end{array}$$

Focus on unrefined expansions.

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True result: We find $G_N^{(m_0, m_1)} \sim O(t^{\max(m_0 N + m_0^2, m_1^2)})$, hence

$$Z_{\infty} = Z_{\text{closed}} \sum_{m_1=0}^{\infty} G_{\infty}^{(0, m_1)}$$

- ◆ Examples of D-instanton corrections: let $G_{m_1} \equiv G_{\infty}^{(0,m_1)}$

$$Z_{\infty} = 1 + 6t^2 - 16t^3 + 72t^4 - 240t^5 + 928t^6 - 3456t^7 + \dots$$

$$Z_{\text{closed}} = 1 + 8t + 40t^2 + 160t^3 + 552t^4 + 1712t^5 + 4896t^6 + 13120t^7 + \dots$$

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Lesson 2. Giant graviton expansions can generalize beyond BPS sectors!

Lesson 3. At finite N , the D-brane contributions to a gravitational path integral are best formulated via category theory.

(Analogous to how contributions over instantons/saddles are formulated as integrals over moduli spaces.)

Lesson 4. Chiral holography works nonperturbatively in N .

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Signals deconfinement transition at

$$1 - 3t - 3t^2 + t^3 = 0 \implies t = 2 - \sqrt{3}$$

Hawking-Page?

Our analysis reproduces this singularity in the bulk.

$$Z_{\infty}^{\text{th}}(t) = Z_{\text{closed}}(-t) \sum_{m_1=0}^{\infty} G_{\infty}^{(0,m_1)}(-t)$$

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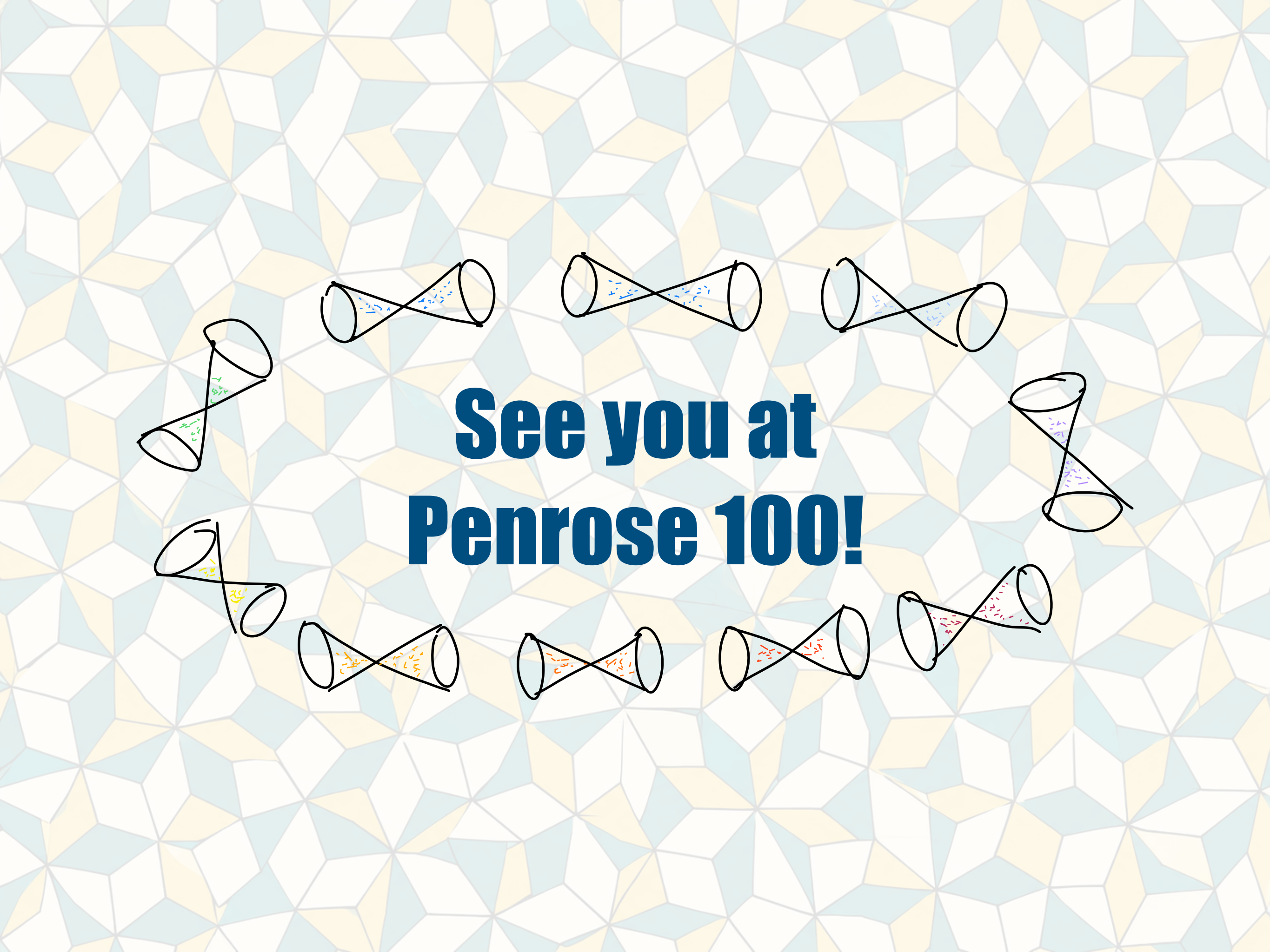
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Lesson 5. Hagedorn singularity reproduced not by closed twistor strings, but instead by the D-instantons!

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Open questions!

- ◆ Relate our bulk $\mathcal{O}_{\mathbb{CP}^3}(-1)^{\oplus 4}$ to twistor space of $\text{AdS}_5 \times S^5$.
- ◆ Derive the detailed holographic dictionary (done for 1/2 BPS sectors).
- ◆ Systematically incorporate small λ corrections using D-instantons.
- ◆ Match correlators — precision non-BPS tests now possible!
- ◆ Interpolate with the strong coupling AdS/CFT duality.



**See you at
Penrose 100!**